Topological insulators and superconductors

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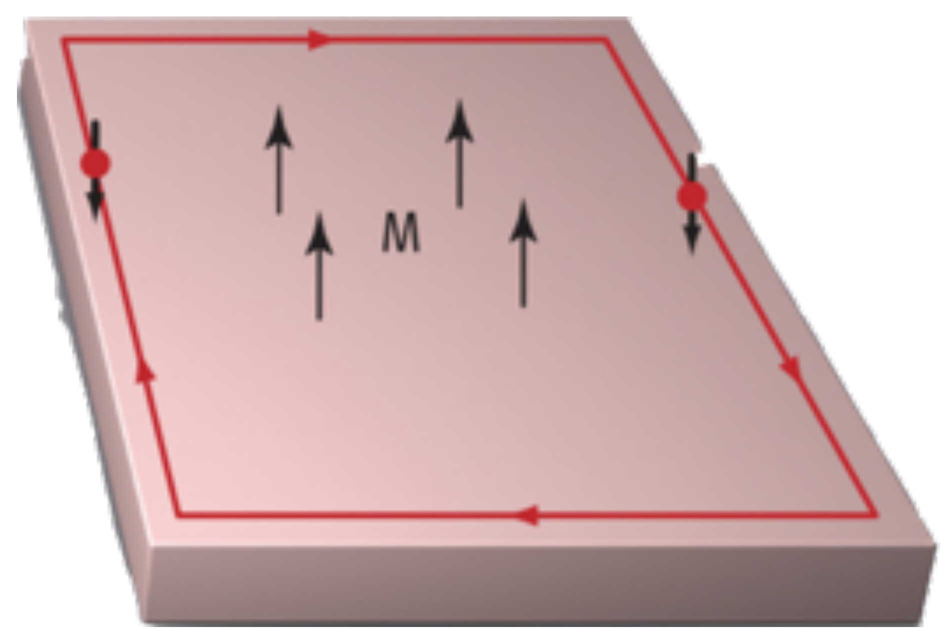
2nd lecture

1. Chern insulator and IQHE
    - Integer quantum Hall effect
    - Chern insulator on square lattice
    - Topological invariant



[picture courtesy S. Zhang et al.]

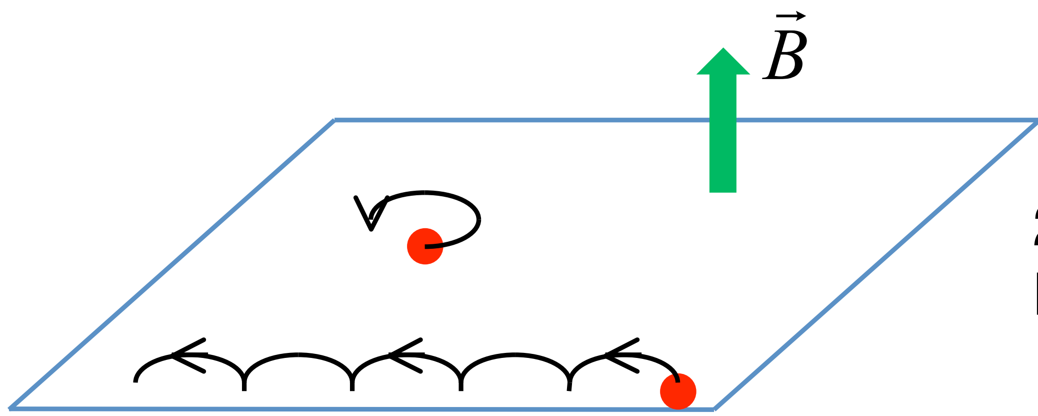
The Integer Quantum Hall State

Integer Quantum Hall State:

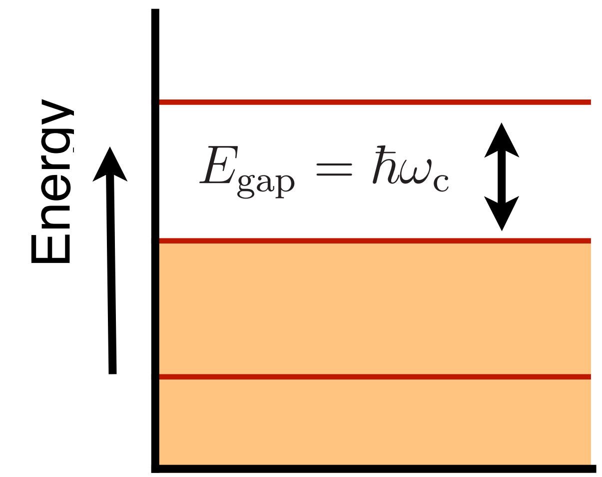
[von Klitzing '80]

First example of 2D topological material

- 2D electron gas in large magnetic field, at low T

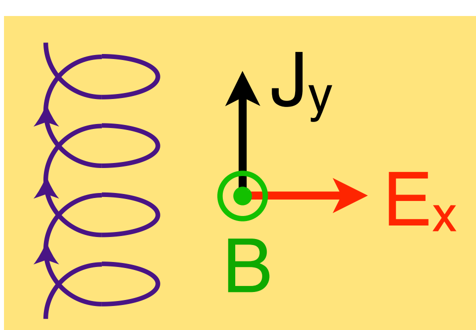


2D cyclotron motion
Landau levels

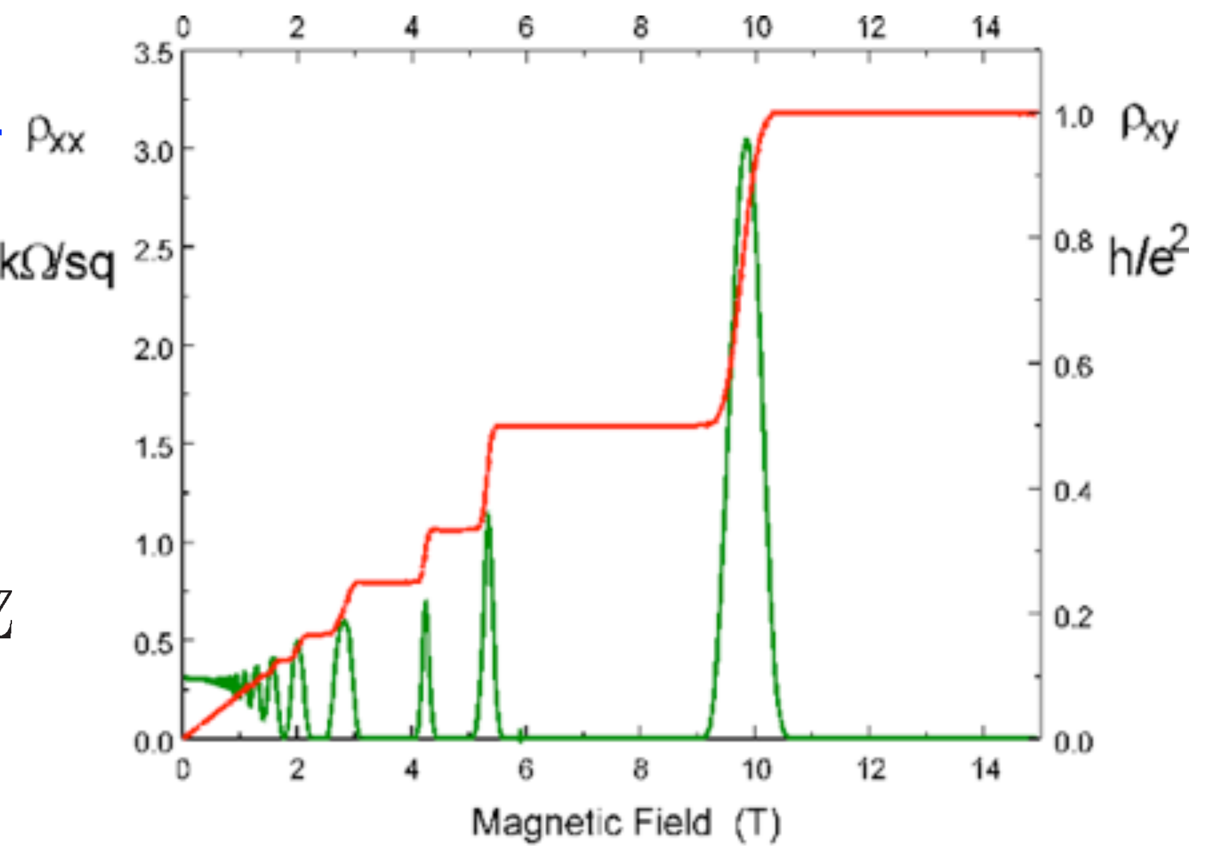


- There is an energy gap, but it is not an insulator

▶ Quantized Hall conductivity: $J_y = \sigma_{xy} E_x$



$$\sigma_{xy} = n\frac{e^2}{h} \qquad n \in \mathbb{Z}$$

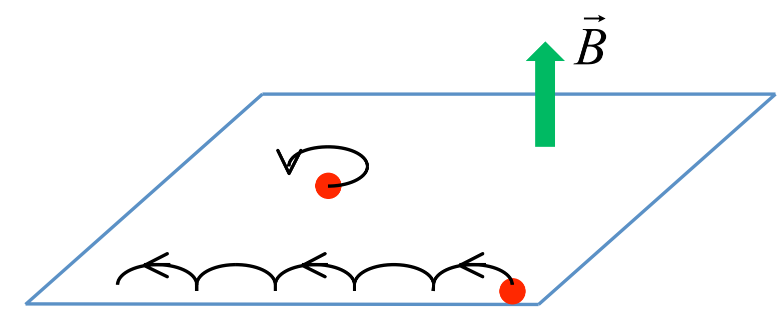


- Plateaus in resistivity

$$\rho_{xy} = \frac{1}{n}\frac{h}{e^2}$$

The Integer Quantum Hall State

What causes the precise quantization in IQHE?



Explanation One: Edge state transport

IQHE has an energy gap in the bulk:

— charge cannot flow in bulk; only along 1D channels at edges (chiral edge states)

— chiral edge state cannot be localized by disorder (no backscattering)

— edge states are perfect charge conductor!

Explanation Two: Topological band theory

Distinction between the integer quantum Hall state and a conventional insulator is a topological property of the band structure ***[Thouless et al, 84]***

$\mathcal{H}(\mathbf{k})$: Brillouin zone $\longmapsto$ Hamiltonians with energy gap

Classified by Chern number: $n = \frac{i}{2\pi} \sum_{\substack{\text{filled} \\ \text{states}}} \int \mathcal{F} d^2 k$ (= topological invariant) $n \in \mathbb{Z}$

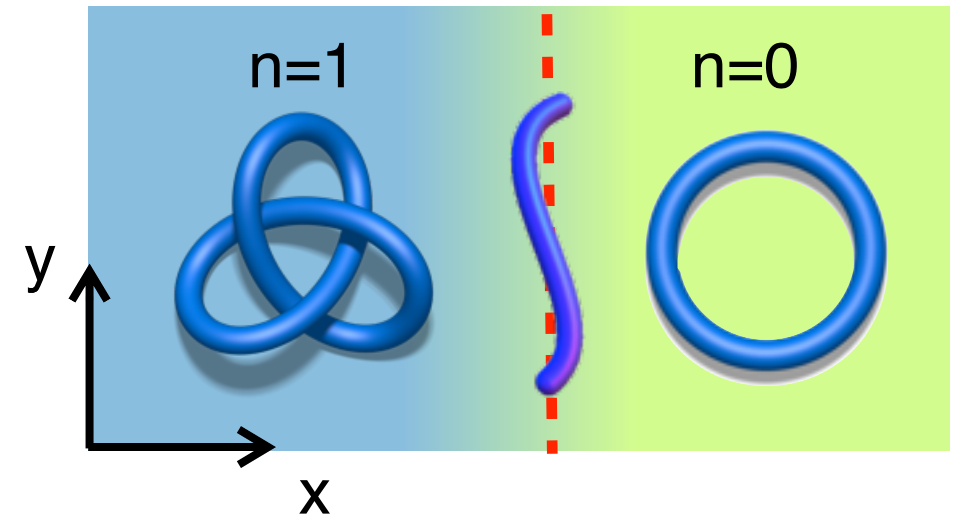
$$\boxed{\text{Kubo formula:} \quad \sigma_{xy} = \frac{e^2}{h} \frac{i}{2\pi} \sum_{\substack{\text{filled} \\ \text{states}}} \int \mathcal{F} d^2 k}$$

$\Longrightarrow$ does not change under smooth deformations, as long as bulk energy gap is not closed

Bulk-boundary correspondence

topological invariant $$n = \frac{i}{2\pi} \sum_{\substack{\text{filled} \\ \text{states}}} \int \mathcal{F} d^2k \qquad n \in \mathbb{Z}$$

Zero-energy state at interface



Bulk-boundary correspondence:

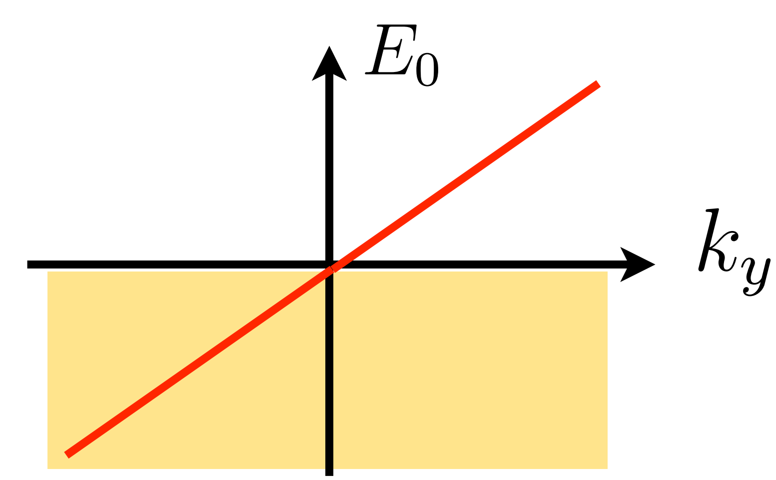
Zero-energy states must exist at the interface between two different topological phases

Follows from the quantization of the topological invariant.

$\Delta n = |n_{\mathrm{L}} - n_{\mathrm{R}}|$ = number or edge modes

Stable gapless edge states:

IQHE: chiral Dirac Fermion



- robust to smooth deformations
  (respect symmetries of the system)
- insensitive to disorder, impossible to localize
- cannot exist in a purely 1D system
  (Fermion doubling theorem)

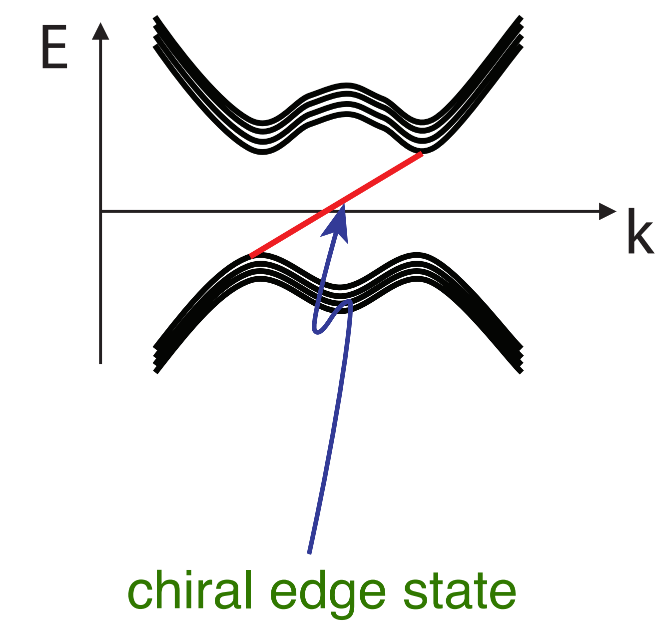
Chern insulator ("integer quantum Hall state on a lattice")

Experimental realization: Cr-doped $(Bi,Sb)_2Te_3$ **[D. Haldane PRL '88] [Chang et al. Science '13]**

Tight-binding model: $H_{\mathrm{CI}} = \begin{pmatrix} c^\dagger_{s,\mathbf{k}} & c^\dagger_{p,\mathbf{k}} \end{pmatrix} \mathcal{H}_{\mathrm{CI}} \begin{pmatrix} c_{s,\mathbf{k}} \\ c_{p,\mathbf{k}} \end{pmatrix}$ $\quad \mathcal{H}_{\mathrm{CI}} = \mathbf{d}(\mathbf{k}) \cdot \vec{\sigma} + \epsilon_0(\mathbf{k})\sigma_0$

$$d_x(\mathbf{k}) = \sin k_x \qquad d_y(\mathbf{k}) = \sin k_y \qquad d_z(\mathbf{k}) = (2 + M - \cos k_x - \cos k_y)$$

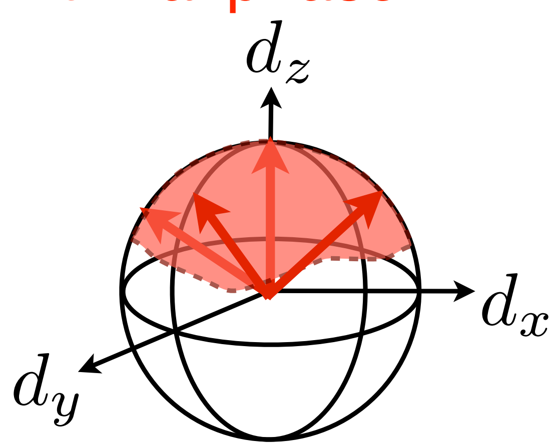
$E_\pm = \pm |\mathbf{d}(\mathbf{k})|$ $\qquad$ Spectrum flattening: $\hat{\mathbf{d}}(\mathbf{k}) = \dfrac{\mathbf{d}(\mathbf{k})}{|\mathbf{d}(\mathbf{k})|}$



trivial phase

$M > 0$

$n_{\mathbb{Z}} = 0$

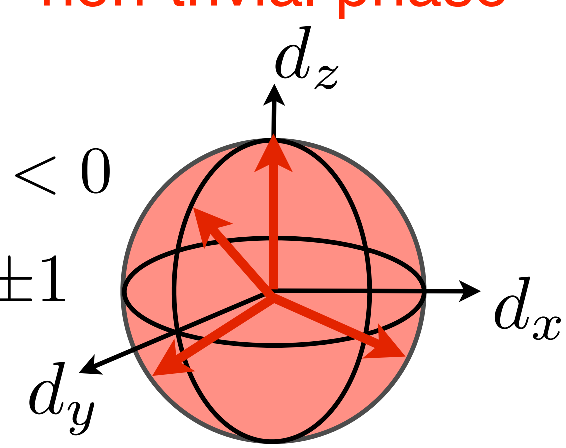


no edge state

non-trivial phase

$-4 < M < 0$

$n_{\mathbb{Z}} = \pm 1$



chiral edge state

Chern number: $n_{\mathbb{Z}} = \dfrac{1}{8\pi} \displaystyle\int_{\mathrm{BZ}} d^2\mathbf{k}\, \epsilon^{\mu\nu} \hat{\mathbf{d}} \cdot \left[\partial_{k_\mu} \hat{\mathbf{d}} \times \partial_{k_\nu} \hat{\mathbf{d}} \right]$ $\quad$ quantized Hall effect $\sigma_{xy} = \dfrac{e^2}{h} n$

Mapping $\quad \hat{\mathbf{d}}(\mathbf{k})$: Brillouin zone $\longmapsto \hat{\mathbf{d}}(\mathbf{k}) \in S^2$ $\qquad$ "$\pi_2(S^2) = \mathbb{Z}$"

Chern insulator ("integer quantum Hall state on a lattice")

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$$d_x(\mathbf{k}) = \sin k_x \qquad d_y(\mathbf{k}) = \sin k_y \qquad d_z(\mathbf{k}) = (2 + M - \cos k_x - \cos k_y)$$

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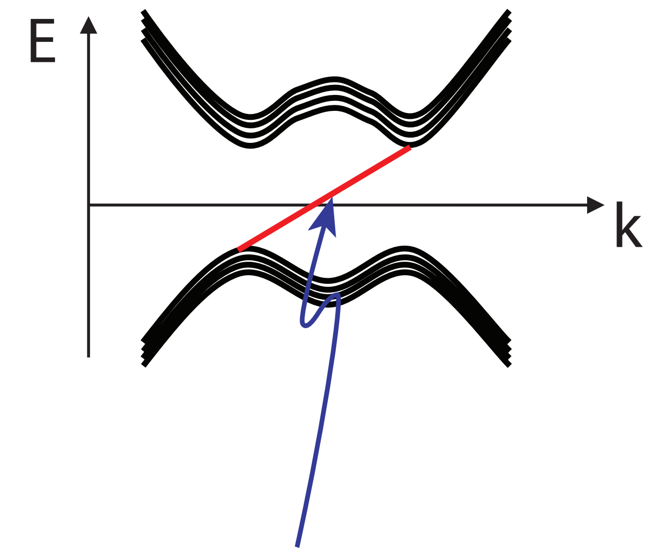
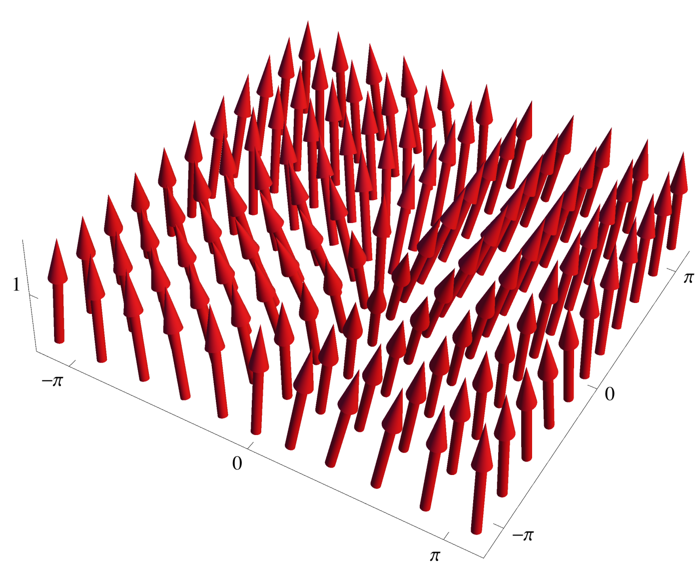


chiral edge state

Texture of unit vector $\hat{\mathbf{d}}(\mathbf{k})$

trivial phase

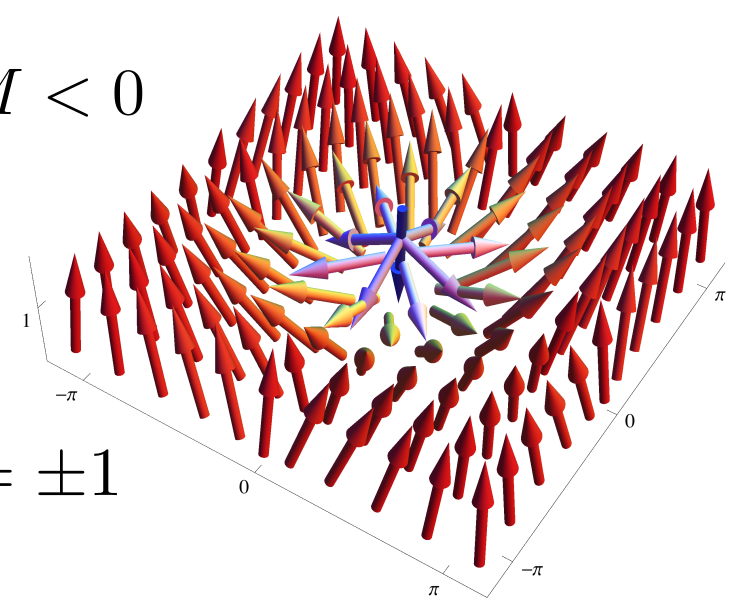
$M > 0$



$n_{\mathbb{Z}} = 0$

non-trivial phase

$-4 < M < 0$



$n_{\mathbb{Z}} = \pm 1$

Chern number: $n_{\mathbb{Z}} = \dfrac{1}{8\pi}\displaystyle\int_{\mathrm{BZ}} d^2\mathbf{k}\, \epsilon^{\mu\nu}\hat{\mathbf{d}} \cdot \left[\partial_{k_\mu}\hat{\mathbf{d}} \times \partial_{k_\nu}\hat{\mathbf{d}}\right]$

Chern insulator on square lattice

Chern insulator on square lattice: $\mathcal{H}_{\mathrm{CI}} = \mathbf{d}(\mathbf{k}) \cdot \vec{\sigma} + \epsilon_0(\mathbf{k})\sigma_0$

$$d_x(\mathbf{k}) = \sin k_x \qquad d_y(\mathbf{k}) = \sin k_y \qquad d_z(\mathbf{k}) = (2 + M - \cos k_x - \cos k_y)$$

Effective low-energy continuum theory for M=0: (expand around $\mathbf{k} = 0$; σ_0 term can be neglected)

$$H_{\mathrm{CI}} = k_x \sigma_x + k_y \sigma_y + M \sigma_z$$

two eigenfunctions with energies: $E_{\pm} = \pm\lambda = \pm\sqrt{\mathbf{k}^2 + M^2}$

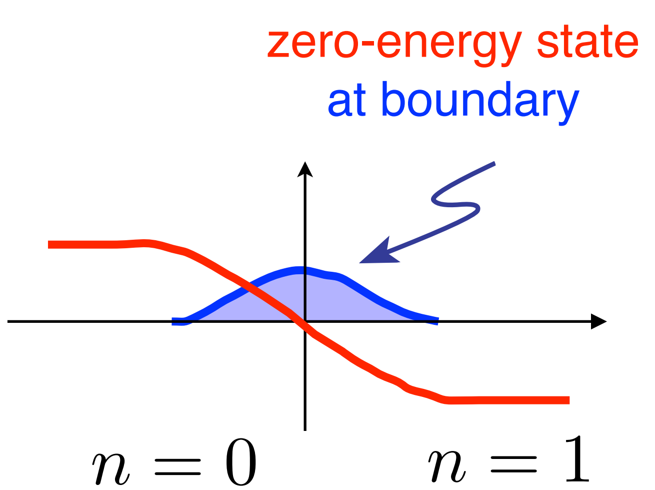
$$|u_{\mathbf{k}}^{+}\rangle = \frac{1}{\sqrt{2\lambda(\lambda - M)}} \begin{pmatrix} k_x - ik_y \\ \lambda - M \end{pmatrix} \qquad |u_{\mathbf{k}}^{-}\rangle = \frac{1}{\sqrt{2\lambda(\lambda + M)}} \begin{pmatrix} -k_x + ik_y \\ \lambda + M \end{pmatrix}$$

Berry curvature: $F_{xy} = \partial_{k_x} A_{k_y} - \partial_{k_y} A_{k_x} = +\frac{M}{2\lambda^3}$

gives nonzero Chern number (= Hall conductance σ_{xy})

$$n = \frac{1}{2\pi} \int d^2k \, F_{xy} = \frac{1}{2}\mathrm{sgn}(M)$$

NB: Chern number must be integer for integrals over compact manifolds. Proper regularization of Dirac Hamiltonian will lead to $n \in \mathbb{Z}$



Chiral edge state at boundary between two Chern insulators with different n

$$\psi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{ik_y y} e^{-\int_0^x M(x')dx'}$$

Experimental realisation of Chern insulator

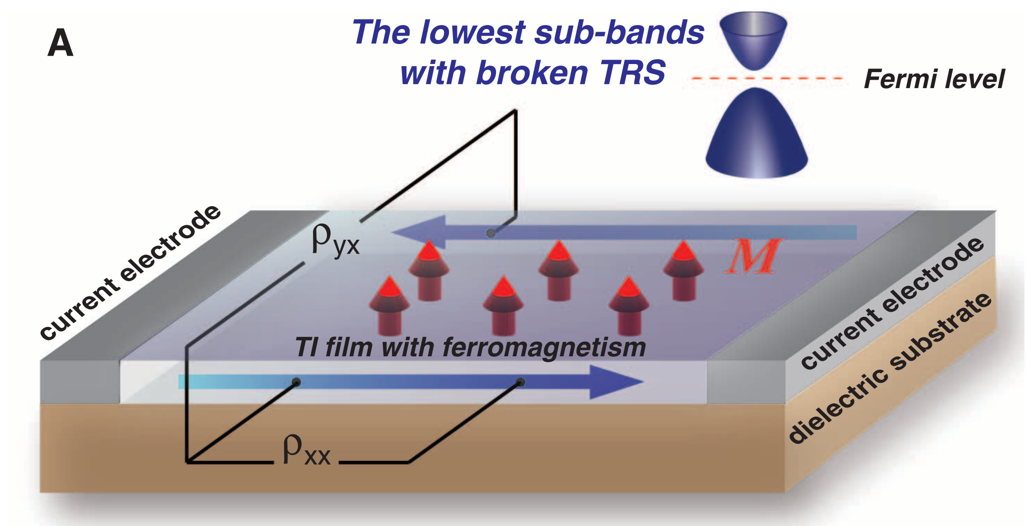
[Chang et al. Science ’13]

- Cr-doped $(Bi,Sb)_2Te_3$

— Thin layer of topological insulator, which has helical surface states

— States on top surface are gapped out by finite size quantization

— Time-reversal symmetry is broken by magnetic ad-atoms (Cr or V)



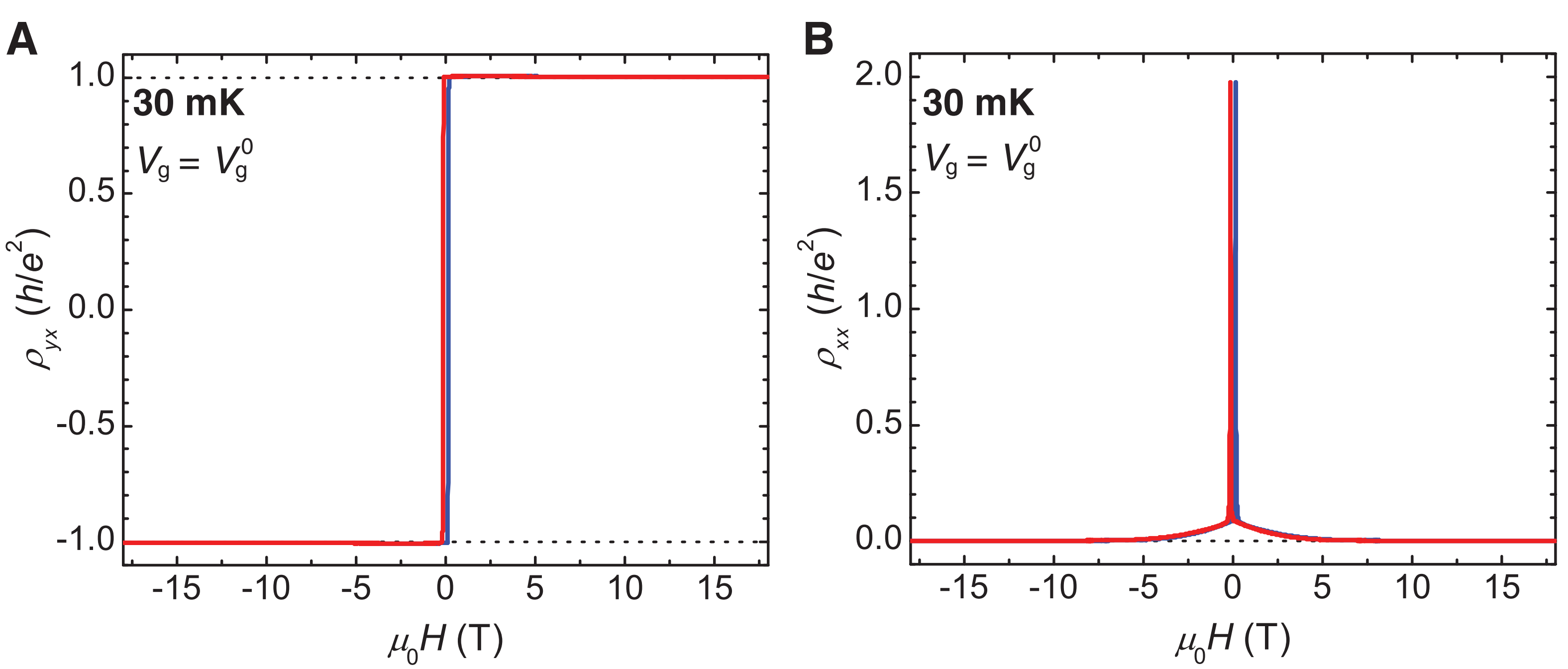


Fig. 3. The QAH effect under strong magnetic field measured at 30 mK. (**A**) Magnetic field dependence of ρ_{yx} at V_g^0. (**B**) Magnetic field dependence of ρ_{xx} at V_g^0. The blue and red lines in (A) and (B) indicate the data taken with increasing and decreasing fields, respectively.