Introduction to topological aspects in condensed matter physics

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3rd lecture

1. Topological superconductors w/ TRS
    - Two-dimensional helical superconductor
    - Three-dimensional TRI topological superconductor
    - Non-centrosymmetric superconductors

2. Periodic table of topological insulators and superconductors
    - Ten-fold way: Symmetry classes
    - Topological classification of non-interacting fermionic systems
    - Bott periodicity

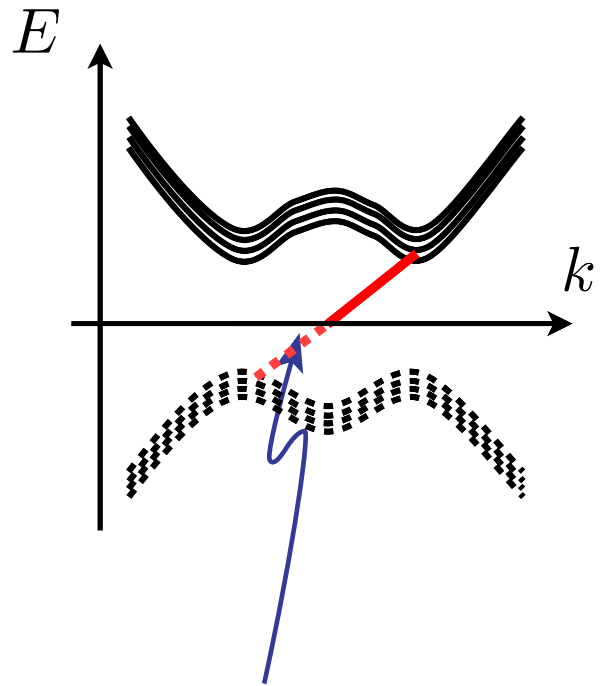
Two-dimensional spinless chiral p-wave SC

Lattice BdG model: **[Read & Green 00]**

$$\mathcal{H}_{\rm BdG} = \left(2t\left[\cos k_x + \cos k_y\right] - \mu\right)\tau_z + \Delta_0\left(\tau_x \sin k_x + \tau_y \sin k_y\right) = \boldsymbol{m}(\boldsymbol{k}) \cdot \boldsymbol{\tau}$$

(similar to Chern insulator, but different symmetries, different physical interpretation)

edge band structure of chiral p-wave SC



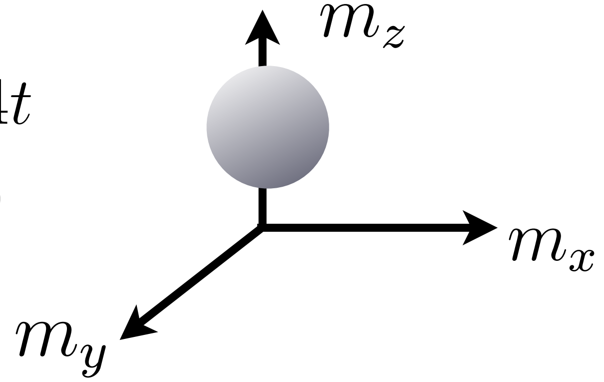
Particle-hole symmetry: $\tau_x \mathcal{H}^*_{\rm BdG}(\mathbf{k})\tau_x = -\mathcal{H}_{\rm BdG}(-\mathbf{k})$

$E = \pm\left|\boldsymbol{m}(\boldsymbol{k})\right|$ Spectrum flattening: $\hat{\boldsymbol{m}}(\boldsymbol{k}) = \dfrac{\boldsymbol{m}(\boldsymbol{k})}{\left|\boldsymbol{m}(\boldsymbol{k})\right|}$

trivial phase

$|\mu| > 4t$

$n = 0$

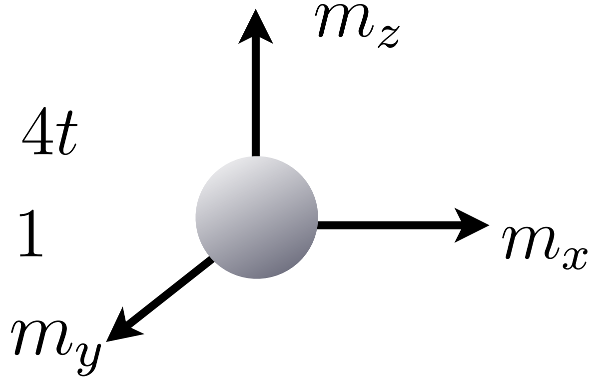


no edge state

non-trivial phase

$|\mu| < 4t$

$n = 1$



chiral Majorana edge state

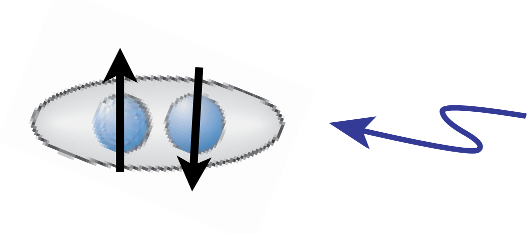
classified by Chern number: (winding number)

$$n = \frac{1}{8\pi}\int_{\rm BZ} d^2\mathbf{k}\,\epsilon^{\mu\nu}\,\hat{\mathbf{m}}\cdot\left[\partial_{k_\mu}\hat{\mathbf{m}} \times \partial_{k_\nu}\hat{\mathbf{m}}\right]$$

Mapping $\hat{\boldsymbol{m}}(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $\hat{\boldsymbol{m}}(\boldsymbol{k}) \in S^2$ "$\pi_2(S^2) = \mathbb{Z}$"

Time-reversal-invariant topological superconductor

Superconducting pairing with spin:


Cooper pair

$$H_{\rm MF} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma\sigma'} \left[\Delta_{\sigma\sigma'}(\mathbf{k}) c^\dagger_{\mathbf{k},\sigma} c^\dagger_{-\mathbf{k},\sigma'} + \Delta^*_{\sigma\sigma'}(\mathbf{k}) c_{-\mathbf{k},\sigma'} c_{\mathbf{k},\sigma} \right]$$

2 x 2 Gap matrix: $\Delta(\mathbf{k}) = [\Delta_s(\mathbf{k})\sigma_0 + \mathbf{d}(\mathbf{k}) \cdot \vec{\sigma}] i\sigma_y$

Time-reversal symmetry: $\sigma_y \Delta^\dagger(\mathbf{k}) \sigma_y = \Delta^{\rm T}(-\mathbf{k})$

Different spin-pairing symmetries: (anti-symmetry of wavefunction)

spin-singlet: $\Delta_s(\mathbf{k}) : \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ even parity: $\Delta_s(\mathbf{k}) = \Delta_s(-\mathbf{k})$

spin-triplet:
$$\begin{cases} d_x(\mathbf{k}) - i d_y(\mathbf{k}) : \ |\uparrow\uparrow\rangle \\ d_x(\mathbf{k}) + i d_y(\mathbf{k}) : \ |\downarrow\downarrow\rangle \\ d_z(\mathbf{k}) : \ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \end{cases}$$
odd parity: $\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$

2D time-reversal-invariant topological superconductor

(also known as “helical superconductor”)

Square lattice BdG Hamiltonian in the presence of time-reversal symmetry:

Simplest model:
(spinless chiral p-wave SC)²

$$\mathcal{H}_{\text{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y) - \mu \qquad d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = 0$$

TRS: $\boxed{\Theta\mathcal{H}_{\text{BdG}}(\mathbf{k})\Theta^{-1} = +\mathcal{H}_{\text{BdG}}(-\mathbf{k})}$ $\quad \Theta = i\sigma_y \otimes \tau_0\mathcal{K} \qquad \Theta^2 = -1$

PHS: $\boxed{\Xi\,\mathcal{H}_{\text{BdG}}(\boldsymbol{k})\,\Xi^{-1} = -\mathcal{H}_{\text{BdG}}(-\boldsymbol{k})}$ $\quad \Xi = \sigma_0 \otimes \tau_x\mathcal{K} \qquad \Xi^2 = +1$

- ▶ Combination of time-reversal and particle-hole symmetry:

(chiral symmetry) $\quad U_S = (i\sigma_y \otimes \tau_0)(\sigma_0 \otimes \tau_x) \qquad U_S\mathcal{H}_{\text{BdG}}(\mathbf{k}) + \mathcal{H}_{\text{BdG}}(\mathbf{k})U_S = 0$

- ▶ $\mathcal{H}_{\text{BdG}}$ can be brought into block-off diagonal form: (transform to basis in which S is diagonal)

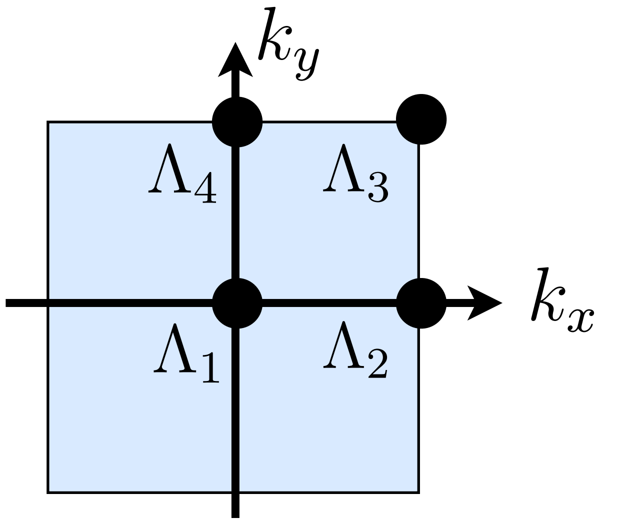
$$\widetilde{\mathcal{H}}_{\text{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \qquad D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\right\}$$

- ▶ TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

2D time-reversal-invariant topological superconductor

$$\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^{\dagger}(\mathbf{k}) & 0 \end{pmatrix}$$

where: $D(\mathbf{k}) = (i\sigma_y)\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}]\}$



- Spectrum flattening:

Projector onto filled Bloch bands

$$Q = \mathbb{1}_{4N} - 2P \qquad Q(\mathbf{k}) = \begin{pmatrix} 0 & q(\mathbf{k}) \\ q^{\dagger}(\mathbf{k}) & 0 \end{pmatrix}$$

- TRS acts on $q(\mathbf{k})$ as follows: $q(\mathbf{k}) = -q^T(-\mathbf{k})$

- The eigenfunctions of $Q(\mathbf{k})$ are:

$$|u_a^{\pm}(\boldsymbol{k})\rangle_{\mathrm{N}} = \frac{1}{\sqrt{2}}\begin{pmatrix} n_a \\ \pm q^{\dagger}(\boldsymbol{k}) n_a \end{pmatrix}$$

where: $(n_a)_b = \delta_{ab}$

are globally defined.

Z_2 topological invariant:

sewing matrix

$$(-1)^{\nu} = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[\omega(\Lambda_a)\right]}{\sqrt{\det\left[\omega(\Lambda_a)\right]}} = \pm 1 \qquad \omega(\mathbf{k}) = {}_{\mathrm{N}}\langle u_a^-(-\mathbf{k})|\Theta u_b^-(\mathbf{k})\rangle_{\mathrm{N}}$$

$$\Rightarrow \boxed{(-1)^{\nu} = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[q^T(\Lambda_a)\right]}{\sqrt{\det\left[q(\Lambda_a)\right]}} = \pm 1}$$

$q(\mathbf{k}) = -q^T(-\mathbf{k})$

$q^{\dagger}(\mathbf{k}) = q^{-1}(\mathbf{k})$

(same symmetries as sewing matrix)

2D time-reversal-invariant topological superconductor

Effective low-energy continuum theory: (expand around $\mathbf{k} = 0$)

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k}) \cdot \vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k}) \cdot \vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

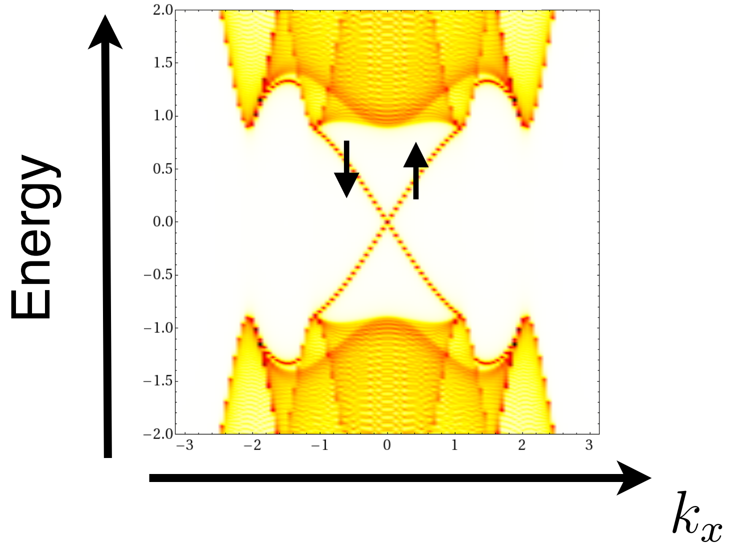
$$\varepsilon(\mathbf{k}) = -tk^2 + 4t - \mu \qquad d_x(\mathbf{k}) = k_x \qquad d_y(\mathbf{k}) = k_y \qquad d_z(\mathbf{k}) = 0$$

Energy spectrum: $E_\pm = \pm\lambda(\mathbf{k}) = \pm\sqrt{\varepsilon^2(\mathbf{k}) + \Delta_t(k_x^2 + k_y^2)}$ TRIM: $k = 0, \quad k = +\infty$

Z_2 topological invariant : $(-1)^\nu = -\mathrm{sgn}(4t - \mu)\mathrm{sgn}(t)$

$\mu > 4t :$ trivial super-conductor

$\mu < 4t :$ TRI topological superconductor



Bulk-boundary correspondence:

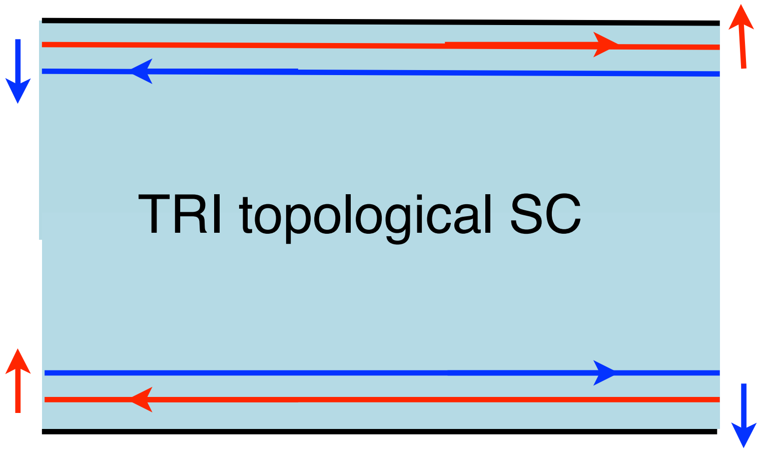
By analogy to chiral p-wave SC: (for $|\mu| < 4t$)
two counter-propagating Majorana edge modes

— protected by TRS and PHS

— two-dimensional analog of B phase of ^{3}He

— possible condensed matter realization:
thin film of $\mathrm{CePt_3Si}$?

helical Majorana edge states:

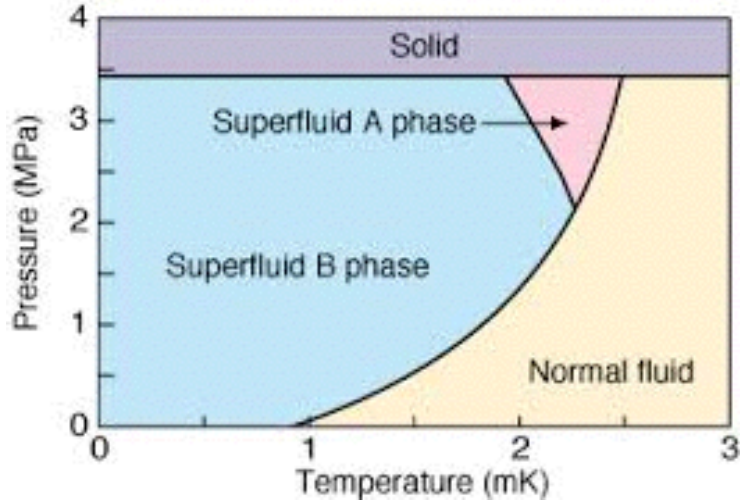


3D time-reversal-invariant topological superconductor

Cubic lattice BdG Hamiltonian in the presence of time-reversal symmetry:

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

equivalent to B phase of ^{3}He



$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y + \cos k_z) - \mu$$

$$d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = \sin k_z$$

TRS: $\boxed{\Theta\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})\Theta^{-1} = +\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})}$ $\quad \Theta = i\sigma_y \otimes \tau_0\mathcal{K} \quad \Theta^2 = -1$

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Chiral symmetry (TRS x PHS): $U_S\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) + \mathcal{H}_{\mathrm{BdG}}(\mathbf{k})U_S = 0$

- ▶ $\mathcal{H}_{\mathrm{BdG}}$ can be brought into block-off diagonal form: (transform to basis in which S is diagonal)

$$\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \qquad D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\right\}$$

- ▶ TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

3D time-reversal-invariant topological superconductor

Lattice BdG Hamiltonian: $\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix}$

▶ Off-diagonal block: $\boxed{D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}]\right\}}$

Mapping $D(\mathbf{k})$: Brillouin zone $\longmapsto$ $D(\mathbf{k})$ TRS: $D(\mathbf{k}) = -D^T(\mathbf{k})$

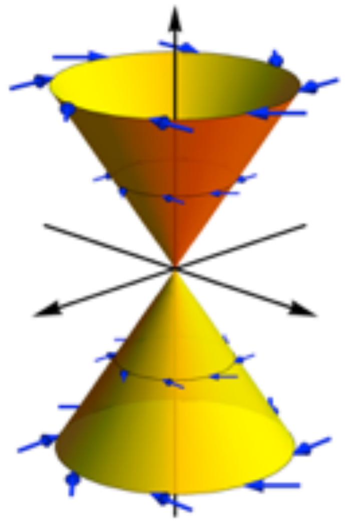
▶ Spectrum flattening: $q(\mathbf{k}) = \sum_a \frac{1}{\lambda_a(\mathbf{k})} u_a(\mathbf{k}) u_a^\dagger(\mathbf{k}) D(\mathbf{k})$ $u_a(\boldsymbol{k})$: eigenvectors of $DD^\dagger$

Mapping $q(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $q(\boldsymbol{k}) \in U(2)$ $\pi_2[U(2)] = 0$

TRS: $q(\mathbf{k}) = -q^T(-\mathbf{k})$ $\pi_3[U(2)] = \mathbb{Z}$

$\Longrightarrow$ classified by winding number: $\boxed{W = \frac{1}{24\pi^2} \int_{\mathrm{BZ}} d^3k\, \varepsilon^{\mu\nu\rho}\, \mathrm{Tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]}$

▶ Bulk-boundary correspondence:

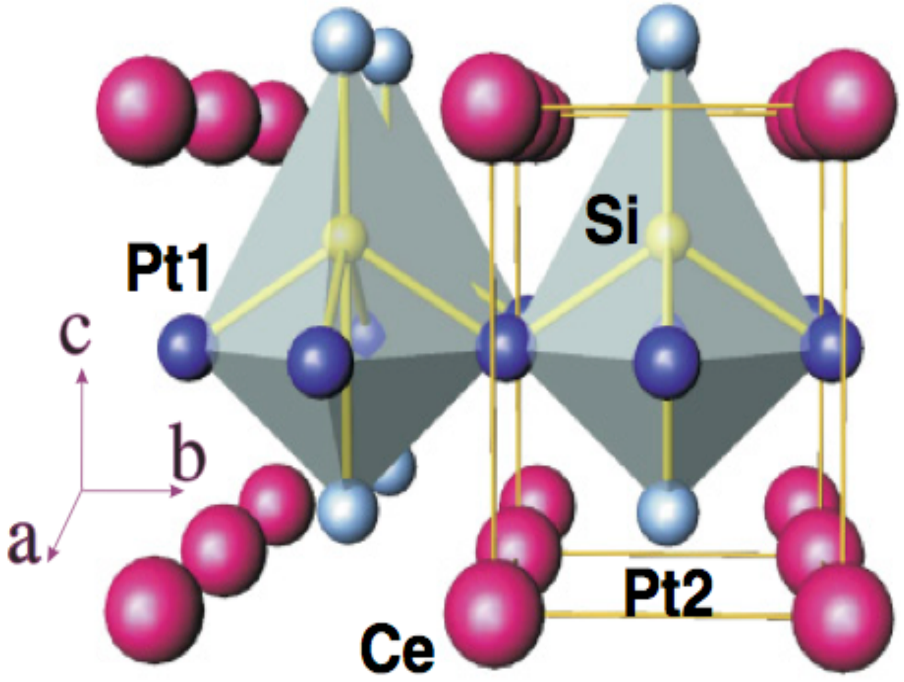


$|W| = \#$ Kramers-degenerate Majorana states

Possible condensed matter realization:
$CePt_3Si$, Li_2Pt_3B, $CeRhSi_3$, $CeIrSi_3$, etc.

Non-centrosymmetric Superconductors

(full gap)



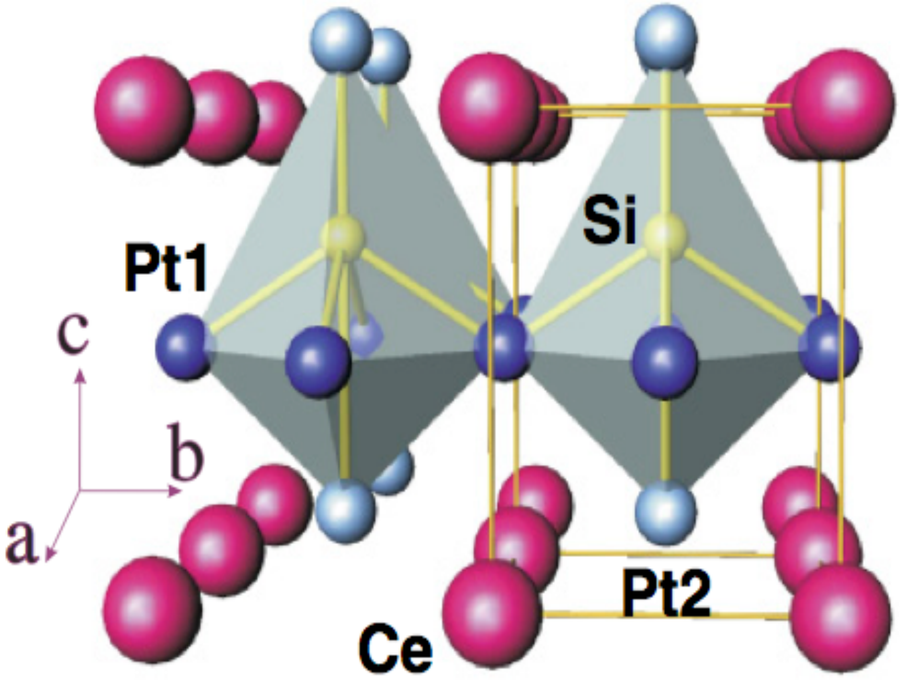
What is a non-centrosymmetric superconductor (NCS)?

Superconductor without a center of inversion in its crystal structure.

$CePt_3Si$, $CeRhSi_3$, $CeIrSi_3$, Li_2Pt_3B, LaPtBi, etc.
Interfaces: $LaAlO_3/SrTiO_3$

[E. Bauer et al, PRL (2004), Kimura et al. ’05, Sugitami et al. ’06, etc..]

Consider tetragonal point group C_{4v}:



Crystal lattice potential gradient

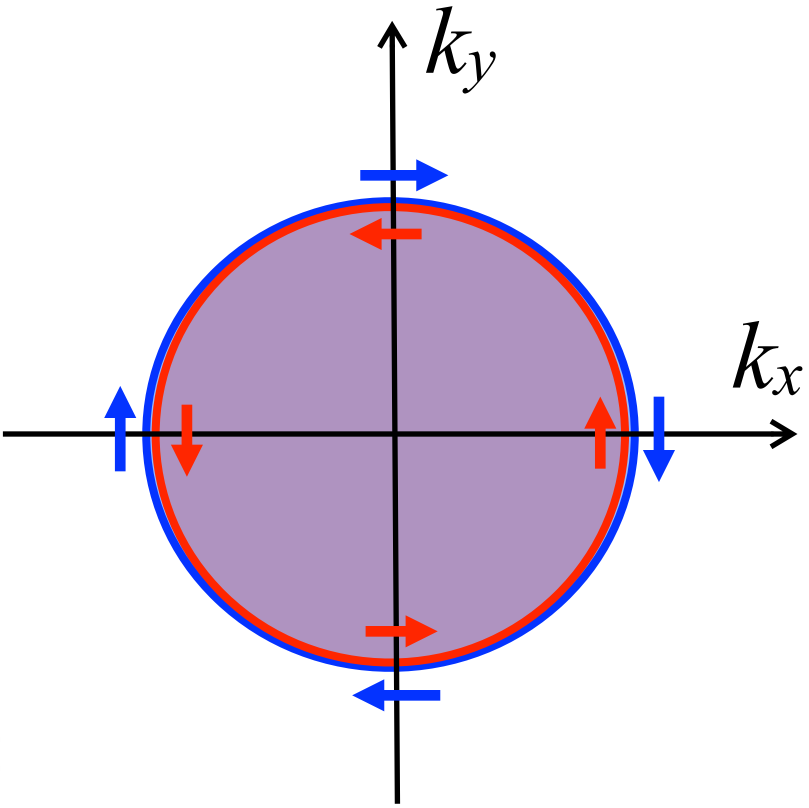
through relativist effects potential gradient leads to anti-symmetric spin-orbit coupling

Non-centrosymmetric SCs: Structure of pairing state

(i) Lack of center of inversion causes anti-symmetric SO coupling.

Normal state: $\mathcal{H} = \sum_{\boldsymbol{k}\mu\nu} c^{\dagger}_{\boldsymbol{k}\mu} \left(\varepsilon_{\boldsymbol{k}} \sigma_0 + \alpha \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} \right)_{\mu\nu} c_{\boldsymbol{k}\nu} = \sum_{\boldsymbol{k}s} \xi_{\boldsymbol{k}s} \, b^{\dagger}_{\boldsymbol{k}s} b_{\boldsymbol{k}s}$

Spin basis: $\mu = \uparrow, \downarrow$ Helicity basis: $s = \pm$



Spin-split energy spectrum:

$$\boxed{\xi^{\pm}_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} \pm |\boldsymbol{g}_{\boldsymbol{k}}|}$$

(ii) Lack of center of inversion allows for admixture of singlet and triplet pairing components

$$\Delta(\boldsymbol{k}) = f(\boldsymbol{k}) \left(\Delta_s \sigma_0 + \Delta_t \boldsymbol{d}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} \right) i\sigma_y$$

$\boldsymbol{d}_{\boldsymbol{k}}$ is constrained by SO interaction: $\boldsymbol{g}_{\boldsymbol{k}} \parallel \boldsymbol{d}_{\boldsymbol{k}}$

Gaps on the two Fermi surfaces: $\boxed{\Delta^{\pm}_{\boldsymbol{k}} = \Delta_s \pm \Delta_t \, |\boldsymbol{d}_{\boldsymbol{k}}|}$

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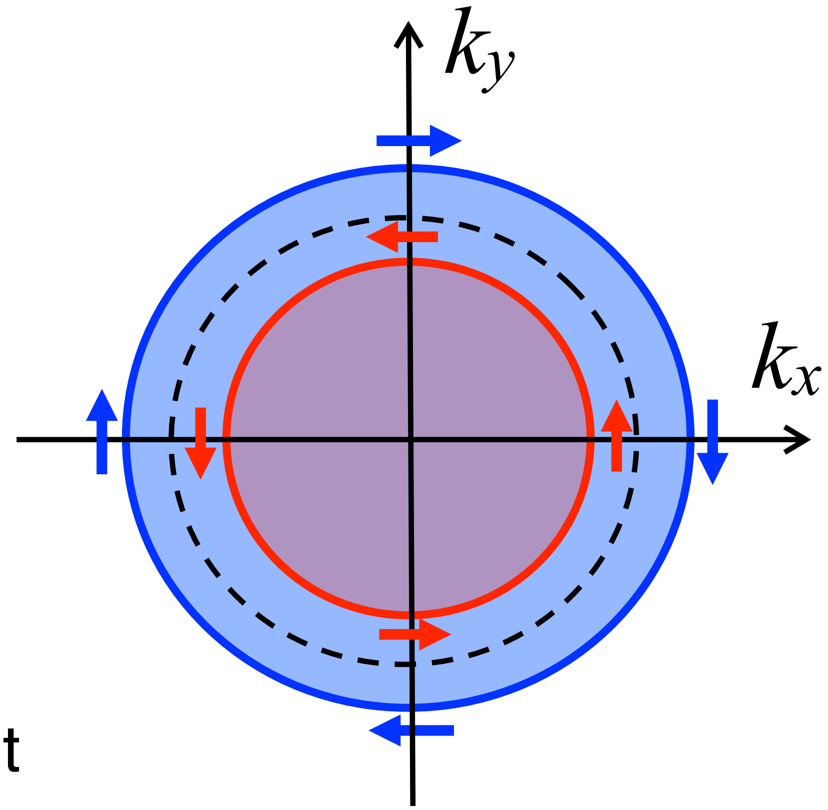

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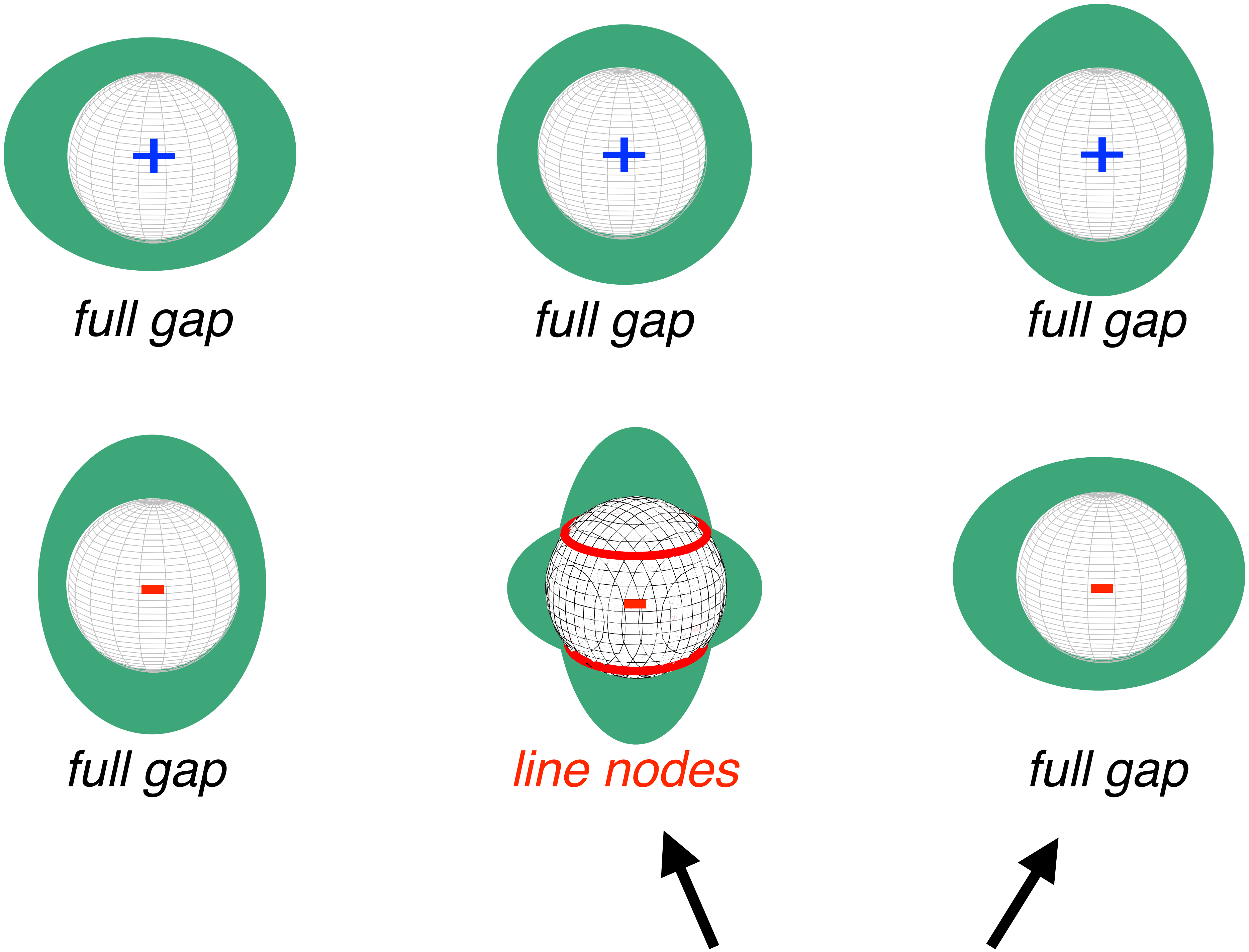
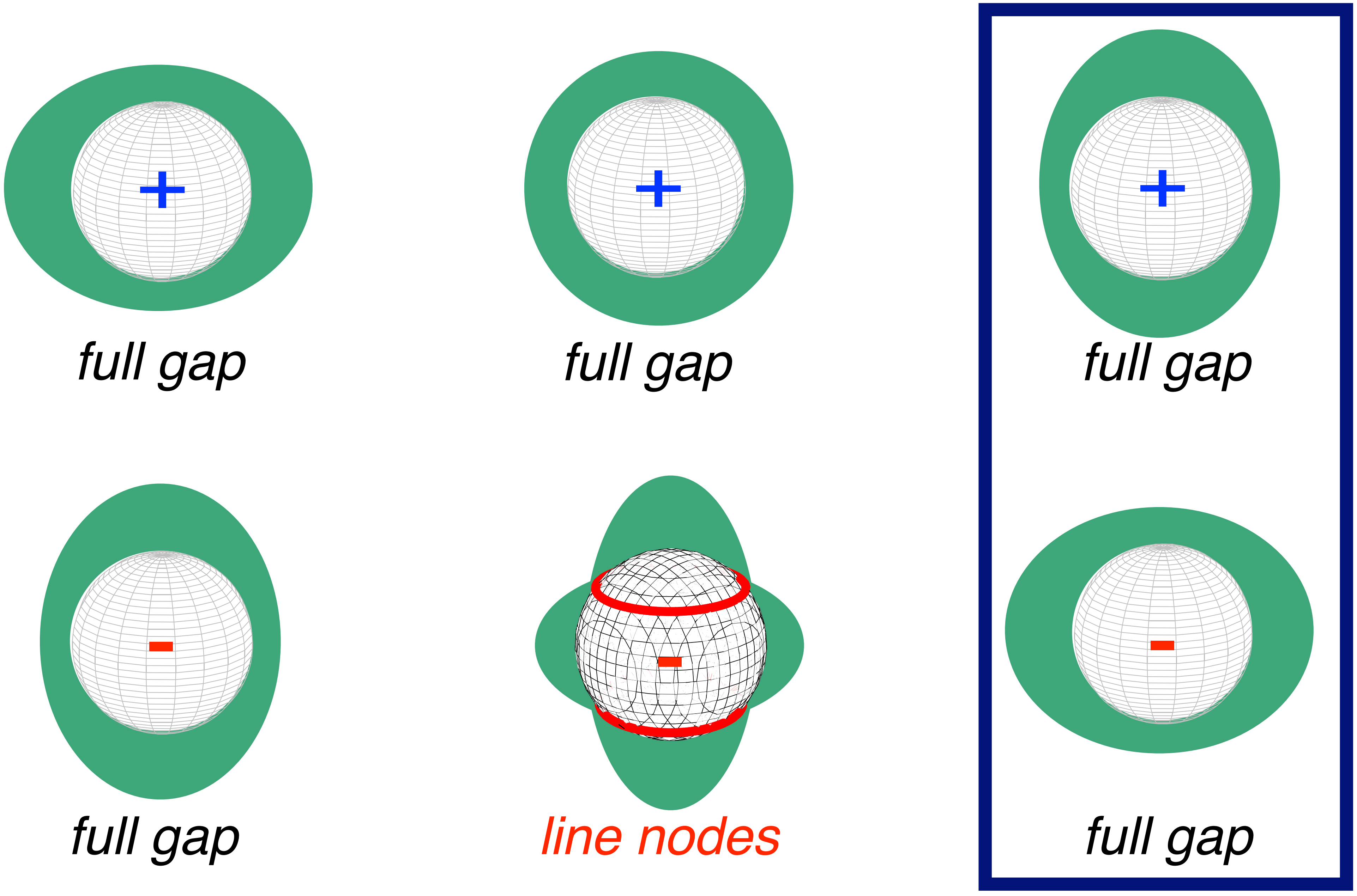
Non-centrosymmetric SCs: Gap structure vs mixing ratio



	$\Delta_s > \Delta_t$	$\Delta_s \sim \Delta_t$	$\Delta_s < \Delta_t$
positive helicity Fermi surface	*full gap*	*full gap*	*full gap*
negative helicity Fermi surface	*full gap*	*line nodes*	*full gap*

Topologically non-trivial

Non-centrosymmetric SCs: Gap structure vs mixing ratio



$\Delta_s > \Delta_t$ | $\Delta_s \sim \Delta_t$ | $\Delta_s < \Delta_t$

positive helicity
Fermi surface

full gap | *full gap* | *full gap*

negative helicity
Fermi surface

full gap | *line nodes* | *full gap*

Topologically non-trivial

Fully gapped non-centrosymmetric superconductor

Lattice BdG Hamiltonian:

$$\mathcal{H}_{\mathrm{BdG}}(\boldsymbol{k}) = \begin{pmatrix} h(\boldsymbol{k}) & \Delta(\boldsymbol{k}) \\ \Delta^{\dagger}(\boldsymbol{k}) & -h^{\mathrm{T}}(-\boldsymbol{k}) \end{pmatrix}$$

- Symmetries:
  - particle-hole symmetry: $\Xi\,\mathcal{H}_{\mathrm{BdG}}(\boldsymbol{k})\,\Xi^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\boldsymbol{k})$
  - time-reversal symmetry: $\Theta\mathcal{H}_{\mathrm{BdG}}(\boldsymbol{k})\Theta^{-1} = +\mathcal{H}_{\mathrm{BdG}}(-\boldsymbol{k})$
  - chiral symmetry (TRS+PHS): $S\mathcal{H}_{\mathrm{BdG}}(\boldsymbol{k}) + \mathcal{H}_{\mathrm{BdG}}(\boldsymbol{k})S = 0$

- $\mathcal{H}_{\mathrm{BdG}}$ can be brought into block-off diagonal form, with off-diagonal component:

$$\boxed{D(\mathbf{k}) = (\varepsilon_{\mathbf{k}} + i\Delta_s)\,\sigma_0 + (\alpha + i\Delta_t)\,\mathbf{l}_{\mathbf{k}} \cdot \vec{\sigma}}$$

Mapping $D(\mathbf{k})$: Brillouin zone $\longmapsto$ $D(\mathbf{k})$ TRS: $D(\mathbf{k}) = -D^{T}(\mathbf{k})$

- Spectrum flattening: $q(\mathbf{k}) = \sum_a \frac{1}{\lambda_a(\mathbf{k})} u_a(\mathbf{k}) u_a^{\dagger}(\mathbf{k}) D(\mathbf{k})$ $u_a(\boldsymbol{k})$: eigenvectors of $DD^{\dagger}$

Mapping $q(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $q(\boldsymbol{k}) \in U(2)$ $\pi_3[U(2)] = \mathbb{Z}$

TRS: $q(\mathbf{k}) = -q^{T}(-\mathbf{k})$ $\pi_2[U(2)] = 0$

Fully gapped non-centrosymmetric superconductor

Lattice BdG Hamiltonian:

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Mapping $q(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $q(\boldsymbol{k}) \in U(2)$ TRS: $q(\mathbf{k}) = -q^{T}(\mathbf{k})$

2D: classified by $\mathbb{Z}_2$ number:

$$\boxed{(-1)^{\nu} = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[q^{T}(\Lambda_a)\right]}{\sqrt{\det\left[q(\Lambda_a)\right]}} = \pm 1}$$

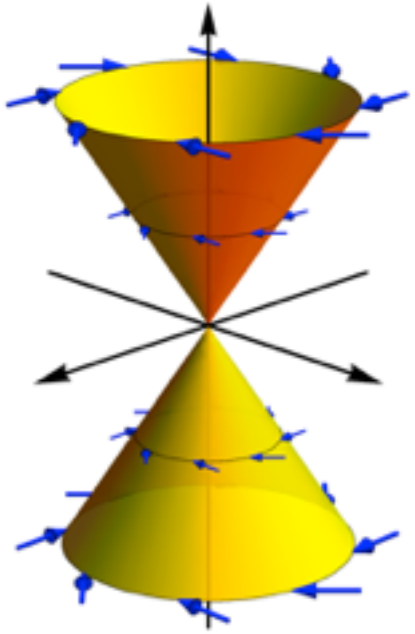
3D: classified by winding number:

$$\boxed{W = \frac{1}{24\pi^2}\int_{\mathrm{BZ}} d^3k\,\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[(q^{-1}\partial_{\mu}q)(q^{-1}\partial_{\nu}q)(q^{-1}\partial_{\rho}q)\right]}$$

Fully gapped non-centrosymmetric SCs: Surface states

2D: $\mathbb{Z}_2$ number ν

3D: winding number W



trivial phase

$$\Delta_t < \Delta_s \ : \begin{cases} \nu = 0 \\ W = 0 \end{cases}$$

no surface state

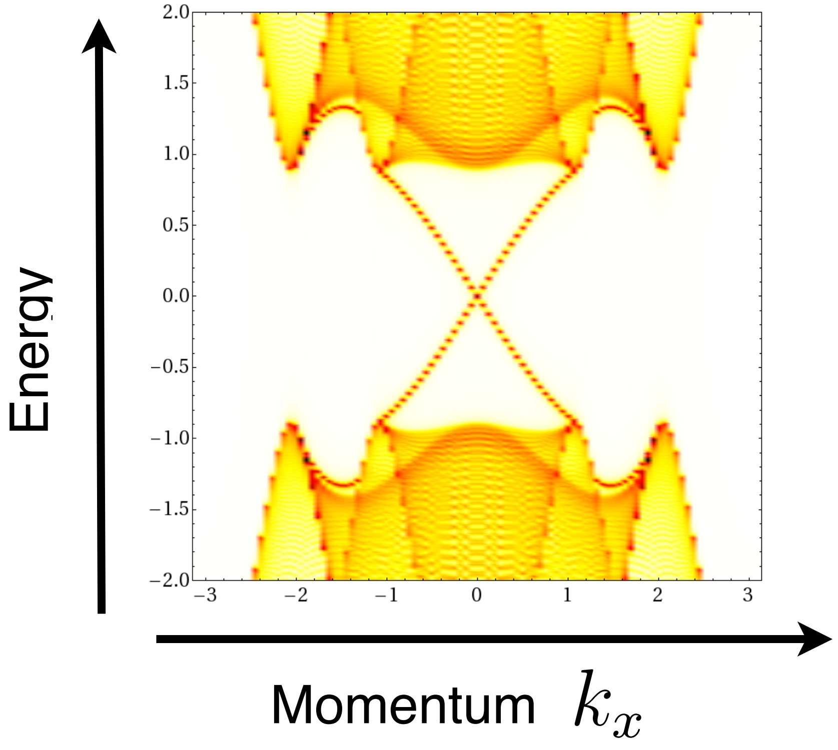
non-trivial phase

$$\Delta_t > \Delta_s \ : \begin{cases} \nu = +1 \\ W = \pm 1, \pm 2, , \cdots \end{cases}$$

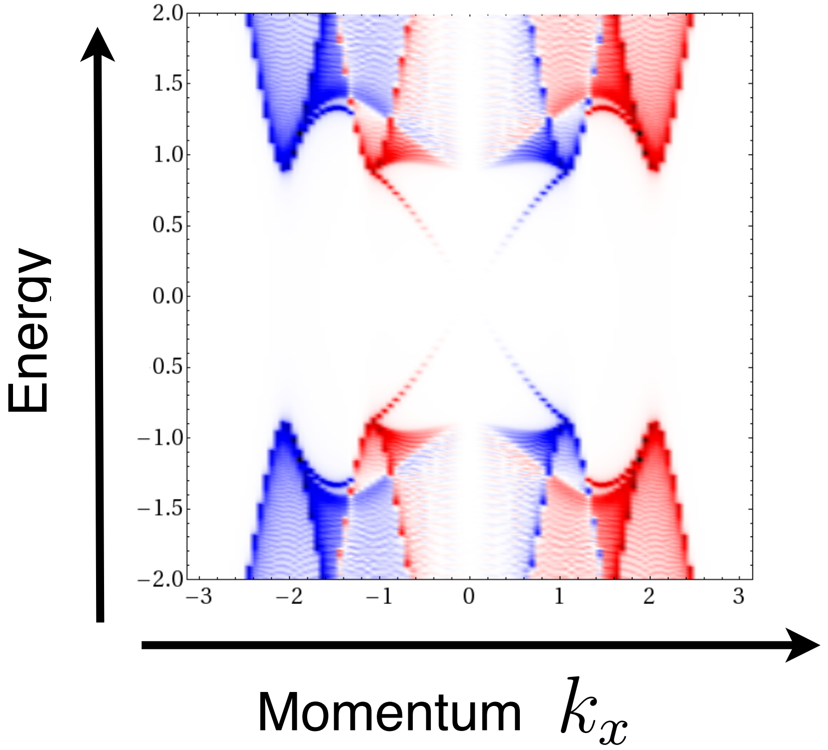
dispersing “helical”
Majorana surface states

Bulk-boundary correspondence: $\nu, \ |W| = \#$ Kramers-degenerate Majorana states

Surface density of states



Y-component of SDOS



Symmetry classes: “Ten-fold way”

(originally introduced in the context of random Hamiltonians / matrices)

time-reversal invariance (Θ): $\Theta = U_T \mathcal{K}$ (is antiunitary)

$$\Theta : \quad U_T \mathcal{H}^*_{\mathrm{BdG}}(\mathbf{k}) U_T^\dagger = +\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})$$ (“reality condition”)

$$\Theta : \begin{cases} 0 & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and} \quad \Theta^2 = +1 \\ -1 & \text{time reversal invariance and} \quad \Theta^2 = -1 \end{cases}$$

complex conjugation

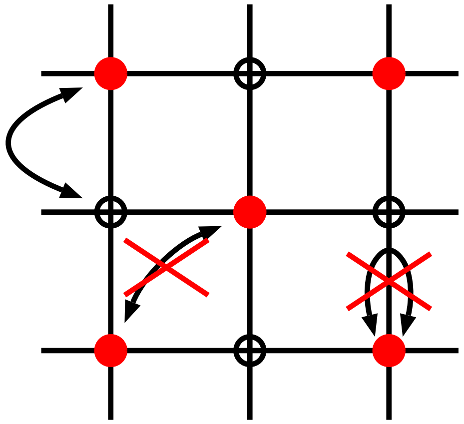
particle-hole symmetry (Ξ): $\Xi = U_C \mathcal{K}$

$$\Xi : \quad U_C \mathcal{H}^*_{\mathrm{BdG}}(\mathbf{k}) U_C^\dagger = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})$$ (“reality condition”)

$$\Xi : \begin{cases} 0 & \text{no particle-hole symmetry} \\ +1 & \text{particle-hole symmetry and} \quad \Xi^2 = +1 \\ -1 & \text{particle-hole symmetry and} \quad \Xi^2 = -1 \end{cases}$$

In addition we can also consider the
“sublattice symmetry” $S = \Theta \Xi$

$$\text{S} : \quad S \mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) + \mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) S = 0$$



Note: SLS is often also called “chiral symmetry”

Classification of topological insulators and superconductors

? Are there more topological insulator and superconductors?

Anti-unitary symmetries:

- time-reversal: $\Theta \mathcal{H}(\mathbf{k}) \Theta^{-1} = +\mathcal{H}(-\mathbf{k}); \quad \Theta^2 = \pm 1$
- particle-hole: $\Xi \mathcal{H}(\mathbf{k}) \Xi^{-1} = -\mathcal{H}(-\mathbf{k}); \quad \Xi^2 = \pm 1$

Chiral (unitary) symmetry: $\Pi \mathcal{H}(\mathbf{k}) \Pi^{-1} = -\mathcal{H}(\mathbf{k}); \quad \Pi \propto \Theta \Xi$

Altland-Zirnbauer Random Matrix Classes

Name	Θ^2	Ξ^2	Π^2
A	0	0	0
AIII	0	0	1
AI	+1	0	0
BDI	+1	+1	1
D	0	+1	0
DIII	-1	+1	1
AII	-1	0	0
CII	-1	-1	1
C	0	-1	0
CI	+1	-1	1

? For which symmetry class and dimension is there a topological insulator/superconductor?

Three different approaches to answer this question:

(i) Anderson localization of boundary modes

Schnyder, Ryu, Furusaki, Ludwig, PRB 2008.

(ii) Dimensional reduction procedures

Qi, Hughes, Zhang, PRB 2008.

Ryu, Schnyder, Furusaki, Ludwig, NJP 2010.

(iii) K-Theory

Kitaev, AIP Conf Proc. 1134, 22 (2009)

All three approaches give the same answer!

Periodic Table of Topological Insulators and Superconductors

Classification of *fully gapped* topological phases

Anti-unitary symmetries:

- time-reversal: $\Theta\mathcal{H}(\mathbf{k})\Theta^{-1} = +\mathcal{H}(-\mathbf{k});\quad \Theta^2 = \pm 1$
- particle-hole: $\Xi\mathcal{H}(\mathbf{k})\Xi^{-1} = -\mathcal{H}(-\mathbf{k});\quad \Xi^2 = \pm 1$

Chiral (unitary) symmetry: $\Pi\mathcal{H}(\mathbf{k})\Pi^{-1} = -\mathcal{H}(\mathbf{k});\quad \Pi \propto \Theta\Xi$

$\mathbb{Z}$: integer classification
$\mathbb{Z}_2$: binary classification
0 : no top. insulator / SC

Altland-Zirnbauer Random Matrix Classes

Name	Θ^2	Ξ^2	Π^2	d=1	d=2	d=3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	0	0	0
BDI	+1	+1	1	$\mathbb{Z}$	0	0
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	$\mathbb{Z}$

Integer Quantum Hall effect

Polyacetylene (assuming SLS)

2D Z_2 topological insulator

3D Z_2 topological insulator

(Only valid for systems with a sufficient number of bands)

Periodic Table of Topological Insulators and Superconductors

→ Classification of *fully gapped* topological phases

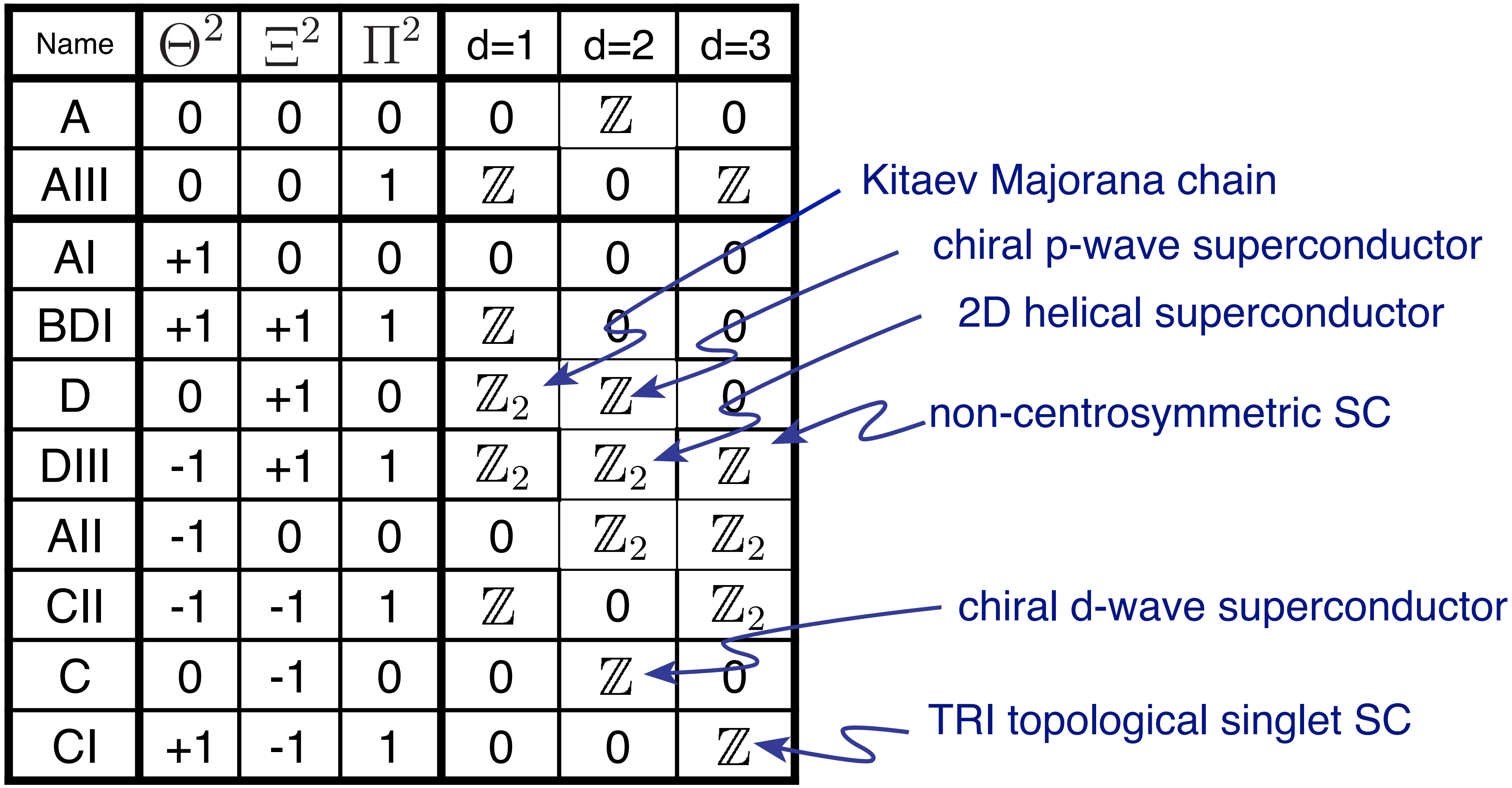
Anti-unitary symmetries:

- time-reversal: $\Theta\mathcal{H}(\mathbf{k})\Theta^{-1} = +\mathcal{H}(-\mathbf{k}); \quad \Theta^2 = \pm 1$
- particle-hole: $\Xi\mathcal{H}(\mathbf{k})\Xi^{-1} = -\mathcal{H}(-\mathbf{k}); \quad \Xi^2 = \pm 1$

Chiral (unitary) symmetry: $\Pi\mathcal{H}(\mathbf{k})\Pi^{-1} = -\mathcal{H}(\mathbf{k}); \quad \Pi \propto \Theta\Xi$

$\mathbb{Z}$: integer classification
$\mathbb{Z}_2$: binary classification
0 : no top. insulator / SC

Altland-Zirnbauer Random Matrix Classes



Name	Θ^2	Ξ^2	Π^2	d=1	d=2	d=3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	0	0	0
BDI	+1	+1	1	$\mathbb{Z}$	0	0
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	$\mathbb{Z}$

Kitaev Majorana chain

chiral p-wave superconductor

2D helical superconductor

non-centrosymmetric SC

chiral d-wave superconductor

TRI topological singlet SC

Periodic Table of Topological Insulators and Superconductors

Classification of *fully gapped* topological phases

Name	Θ^2	Ξ^2	Π^2	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	+1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Altland-Zirnbauer Random Matrix Classes

“complex” K-theory

“real” K-theory (KR-theory)

- Topological invariants: Chern numbers and winding numbers

$$Ch_{n+1}[\mathcal{F}] = \frac{1}{(n+1)!} \int_{\mathrm{BZ}^{d=2n+2}} \mathrm{tr} \left(\frac{i\mathcal{F}}{2\pi} \right)^{n+1}$$

$$\nu_{2n+1}[q] = \frac{(-1)^n n!}{(2n+1)!} \left(\frac{i}{2\pi} \right)^{n+1} \int_{\mathrm{BZ}} \epsilon^{\alpha_1 \alpha_2 \cdots} \mathrm{tr} \left[q^{-1} \partial_{\alpha_1} q \cdot q^{-1} \partial_{\alpha_2} q \cdots \right] d^{2n+1} k$$