Introduction to topological aspects in condensed matter physics

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June 11-13, 2014

Université de Lorraine

4th lecture

1. Beyond ten-fold classification

- Topological classification of non-interacting fermionic systems
- Weak topological insulators and superconductors
- Classification of zero-modes at defects
- Topological crystalline insulators (reflection symmetries)

2. Gapless topological materials

- Examples of topological semi-metals and nodal SCs
- Classification of semi-metals and nodal superconductors
- Example: Nodal non-centrosymmetric SC

Symmetry classes: "Ten-fold way"

(originally introduced in the context of random Hamiltonians / matrices)

time-reversal invariance (Θ): $\Theta = U_T \mathcal{K}$ (is antiunitary)

Θ : $U_T \mathcal{H}^*_{\mathrm{BdG}}(\mathbf{k}) U_T^\dagger = +\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})$ ("reality condition")

$$\Theta : \begin{cases} 0 & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and} \quad \Theta^2 = +1 \\ -1 & \text{time reversal invariance and} \quad \Theta^2 = -1 \end{cases}$$

complex conjugation

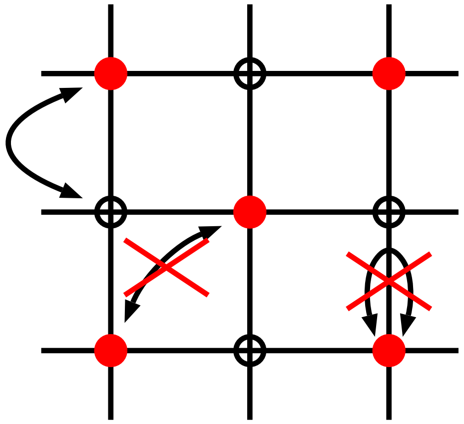
particle-hole symmetry (Ξ): $\Xi = U_C \mathcal{K}$

Ξ : $U_C \mathcal{H}^*_{\mathrm{BdG}}(\mathbf{k}) U_C^\dagger = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})$ ("reality condition")

$$\Xi : \begin{cases} 0 & \text{no particle-hole symmetry} \\ +1 & \text{particle-hole symmetry and} \quad \Xi^2 = +1 \\ -1 & \text{particle-hole symmetry and} \quad \Xi^2 = -1 \end{cases}$$

In addition we can also consider the "sublattice symmetry" $S = \Theta\Xi$

S : $S\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) + \mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) S = 0$



Note: SLS is often also called "chiral symmetry"

Periodic Table of Topological Insulators and Superconductors

Classification of *fully gapped* topological phases

Anti-unitary symmetries:

- time-reversal: $\Theta\mathcal{H}(\mathbf{k})\Theta^{-1} = +\mathcal{H}(-\mathbf{k});\quad \Theta^2 = \pm 1$
- particle-hole: $\Xi\mathcal{H}(\mathbf{k})\Xi^{-1} = -\mathcal{H}(-\mathbf{k});\quad \Xi^2 = \pm 1$

Chiral (unitary) symmetry: $\Pi\mathcal{H}(\mathbf{k})\Pi^{-1} = -\mathcal{H}(\mathbf{k});\quad \Pi \propto \Theta\Xi$

$\mathbb{Z}$: integer classification
$\mathbb{Z}_2$: binary classification
0 : no top. insulator / SC

Altland-Zirnbauer Random Matrix Classes

Name	Θ^2	Ξ^2	Π^2	d=1	d=2	d=3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	0	0	0
BDI	+1	+1	1	$\mathbb{Z}$	0	0
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	$\mathbb{Z}$

Integer Quantum Hall effect

Polyacetylene (assuming SLS)

2D Z_2 topological insulator

3D Z_2 topological insulator

(Only valid for systems with a sufficient number of bands)

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CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
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CI	+1	-1	1	0	0	$\mathbb{Z}$

Kitaev Majorana chain

chiral p-wave superconductor

2D helical superconductor

non-centrosymmetric SC

chiral d-wave superconductor

TRI topological singlet SC

Extension I: Weak topological insulators and supercondutors

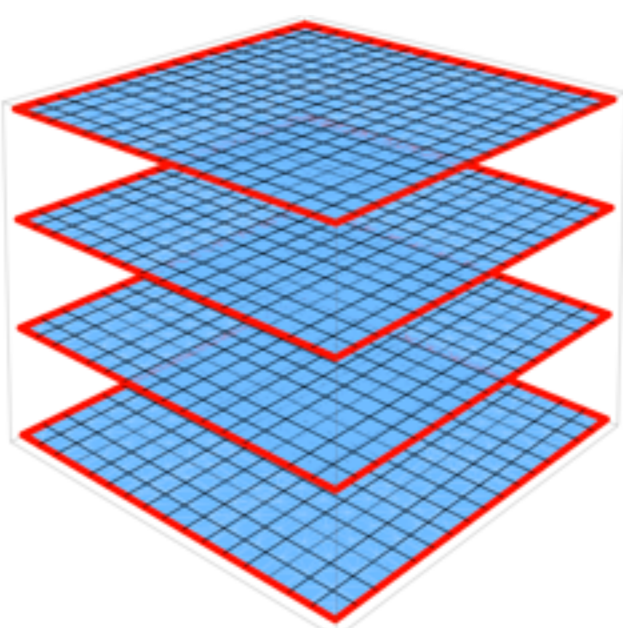
- strong topological insulators (superconductors):
  not destroyed by positional disorder

- weak topological insulators (superconductors):
  only possess topological features
  when translational symmetry is present

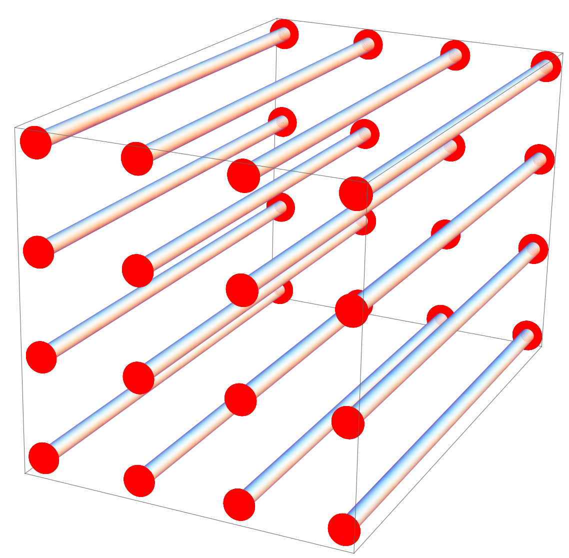
→ weak topological insulators (superconductors) are topologically equivalent to parallel stacks of lower-dimensional strong topological insulator (SCs).

	Symmetry			Dimension			
AZ	T	C	S	1	2	3	4
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	1	-1	1	0	0	$\mathbb{Z}$	0

co-dimension k=1



co-dimension k=2



→ d-dim.weak topological insulators (SCs) of co-dimension k can occur whenever there exists a strong topological state in same symmetry class but in (d-k) dimensions.

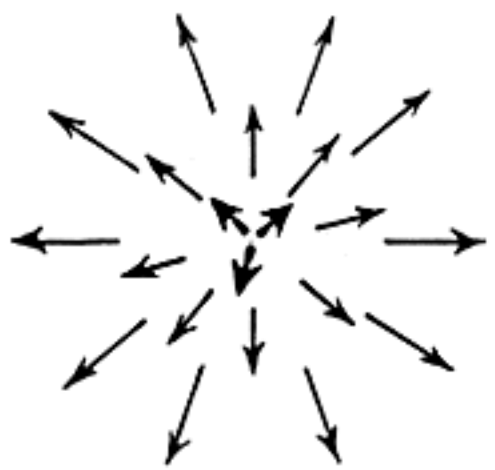
$\binom{d}{k}$ top. invariants $\quad 0 < k \leq d$

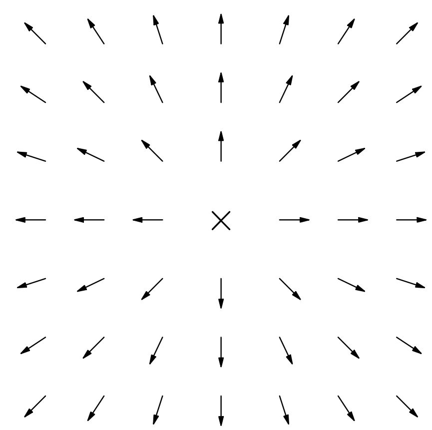
cf. Kitaev, AIP Conf Proc. 1134, 22 (2009)

Extension II: Zero mode localized on topological defect

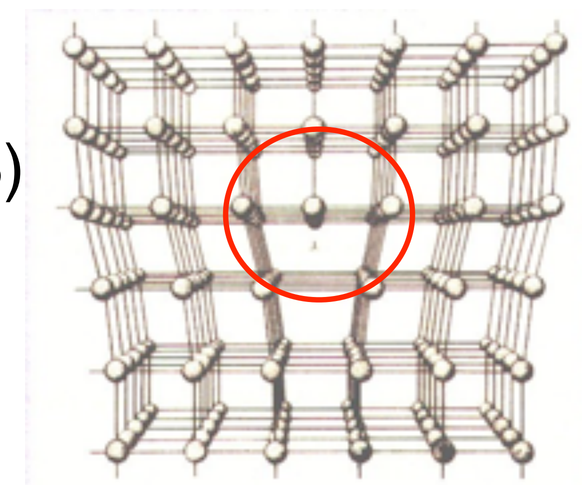
Protected zero modes can also occur at topological defects in D-dim systems

- Point defect (r=0): Hedgehog (D=3), vortex (D=2), domain wall (D=1)





- Line defect (r=1): dislocation line (D=3) domain wall (D=2)



- Two-dim defects (r=2): domain wall (D=3)

Freedman, et. al., PRB (2010)

Teo & Kane, PRB (2010)

Ryu, et al. NJP (2010)

	Symmetry			Dimension			
AZ	T	C	S	1	2	3	4
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	1	-1	1	0	0	$\mathbb{Z}$	0

Can an r-dimensional topological defect of a given symmetry class bind gapless states or not?

→ look at column d=(r+1)

(answer does not depend on D!)

line defect in class A:

$$n = \frac{1}{8\pi^2}\int_{T^3\times S^1} \mathrm{Tr}[\mathcal{F}\wedge\mathcal{F}]$$

(second Chern no = no of zero modes)

Topological Crystalline Insulators

Topological crystalline insulators

Topological insulator protected by global symmetry & mirror symmetry

— consider, e.g., TRS insulator with mirror symmetry **[Teo, Fu, Kane, 2008]**

$$\mathcal{H} = \varepsilon(\mathbf{k})\sigma_z + v(\sin k_x s_y - \sin k_y s_x) \otimes \sigma_x + v_z \sin k_z \sigma_y$$

$(s_i : \text{ spin}; \quad \sigma_i : \text{ orbitals})$

Time-reversal symmetry (class AII): $(is_y)\mathcal{H}^*(\mathbf{k})(is_y)^{-1} = +\mathcal{H}(-\mathbf{k}) \qquad \Theta = is_y\mathcal{K}$

Reflection symmetry: $R = s_x, \;\; R^{-1}\mathcal{H}(-k_x, k_y, k_z)R = \mathcal{H}(k_x, k_y, k_z)$

$$\Longrightarrow \Theta R = -R\Theta$$

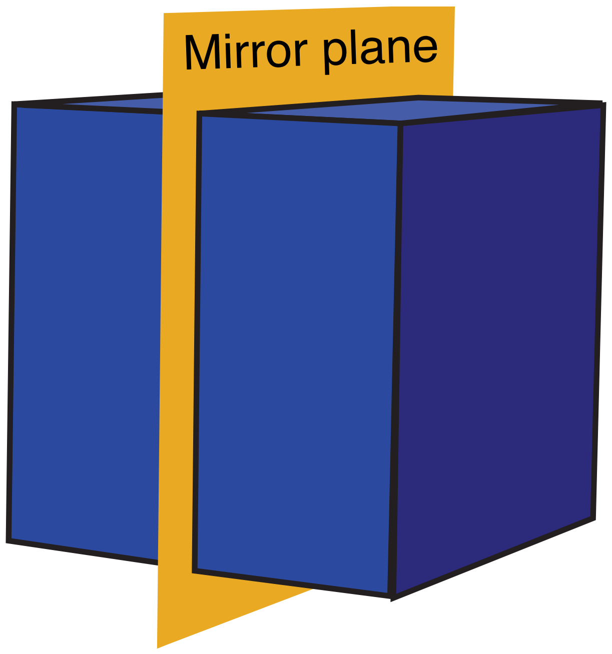
$$\Longrightarrow \mathcal{H}(0, k_y, k_z)R = -R\mathcal{H}(0, k_y, k_z)$$

— define **mirror Chern number** for *each eigenspace of* R: $(\text{ i.e., } s_x = \pm 1)$

$$\mathcal{H}_\pm = \varepsilon_\mathbf{k}\sigma_z \mp v \sin k_y \sigma_x + v_z \sin k_z \sigma_y = \mathbf{m}_\pm(\mathbf{k}) \cdot \boldsymbol{\sigma}$$

$$\Longrightarrow \text{ class D: } \sigma_x \mathcal{H}^*_\pm(\mathbf{k})\sigma_x = -\mathcal{H}(-\mathbf{k})$$

$$\Longrightarrow n_\pm = \frac{1}{8\pi}\int_{\text{2D BZ}} d^2\mathbf{k}\, \epsilon^{\mu\nu}\, \hat{\mathbf{m}}_\pm \cdot \left[\partial_{k_\mu}\hat{\mathbf{m}}_\pm \times \partial_{k_\nu}\hat{\mathbf{m}}_\pm\right]$$

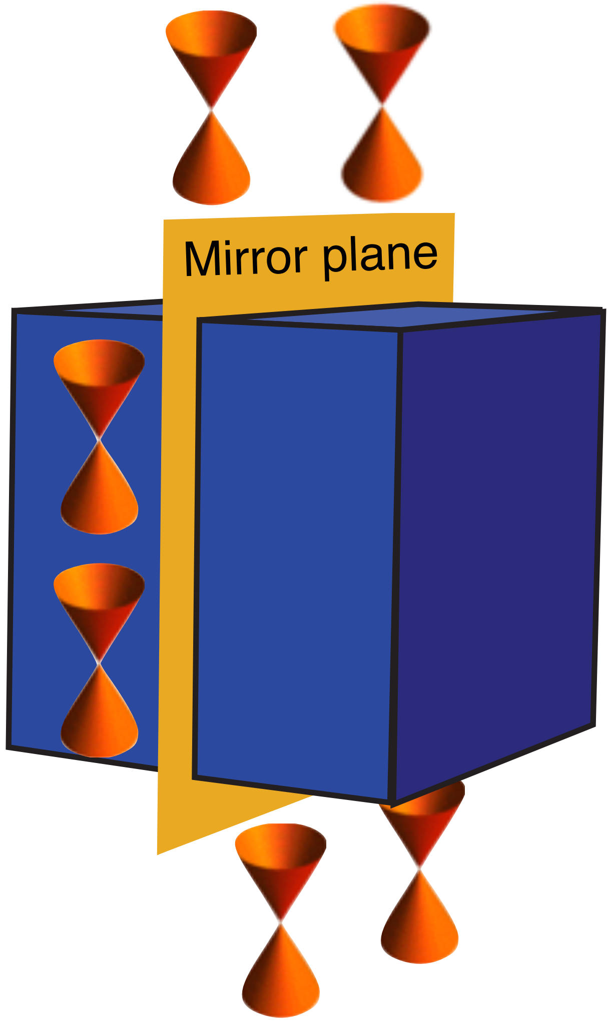


mirror Chern number: $n_\mathcal{M} = (n_+ - n_-)/2$

Topological crystalline insulators

$$n_{\pm} = \frac{1}{8\pi} \int_{\text{2D BZ}} d^2\mathbf{k}\, \epsilon^{\mu\nu} \hat{\mathbf{m}}_{\pm} \cdot \left[\partial_{k_\mu} \hat{\mathbf{m}}_{\pm} \times \partial_{k_\nu} \hat{\mathbf{m}}_{\pm} \right]$$

mirror Chern number: $n_{\mathcal{M}} = n_+ - n_-$

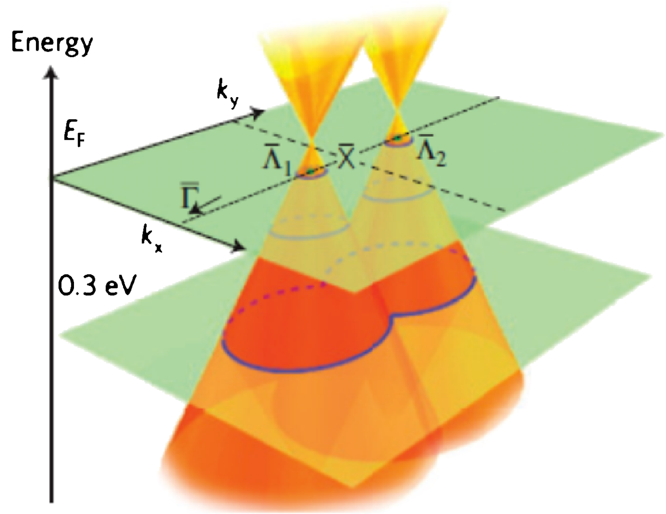


▶ Bulk-boundary correspondence for surfaces that are perpendicular to mirror plane:

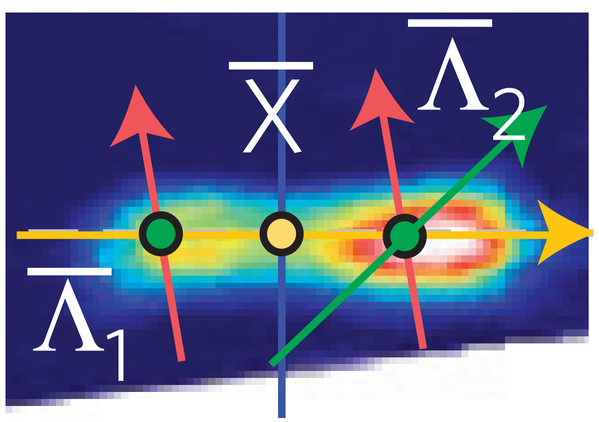
$$n_{\mathcal{M}} = \#$$ Dirac cones surface states

$\Longrightarrow$ two Dirac cones on surface protected by mirror and time-reversal symmetries

▶ SnTe and $Pb_{1-x}Sn_xTe$ are topological crystalline insulators:



ARPES on SnTe



Tanaka, Ando, et al., 2012, 2013

Classification of *fully gapped* topological crystalline materials

Classification in terms of mirror symmetries

$R_- \ : \ R$ anti-commutes with T (C or S)

$R_+ \ : \ R$ commutes with T (C or S)

Reflection	top. insul. and top. SC	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8
R	A	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0
R_+	AIII	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$
R_-	AIII	$M\mathbb{Z}\oplus\mathbb{Z}$	0	$M\mathbb{Z}\oplus\mathbb{Z}$	0	$M\mathbb{Z}\oplus\mathbb{Z}$	0	$M\mathbb{Z}\oplus\mathbb{Z}$	0
R_+,R_{++}	AI	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$
	BDI	$M\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	$M\mathbb{Z}_2$
	D	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0
	DIII	0	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$
	AII	$2M\mathbb{Z}$	0	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0
	CII	0	$2M\mathbb{Z}$	0	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$	$M\mathbb{Z}$	0	0
	C	0	0	$2M\mathbb{Z}$	0	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$	$M\mathbb{Z}$	0
	CI	0	0	0	$2M\mathbb{Z}$	0	$M\mathbb{Z}_2$	$M\mathbb{Z}_2$	$M\mathbb{Z}$
R_-,R_{--}	AI	0	0	$2M\mathbb{Z}$	0	$T\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0
	BDI	0	0	0	$2M\mathbb{Z}$	0	$T\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$
	D	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	$T\mathbb{Z}_2$	$\mathbb{Z}_2$
	DIII	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	$T\mathbb{Z}_2$
	AII	$T\mathbb{Z}_2$	$\mathbb{Z}_2$	$\boxed{M\mathbb{Z}}$	0	0	0	$2M\mathbb{Z}$	0
	CII	0	$T\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$
	C	$2M\mathbb{Z}$	0	$T\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0
	CI	0	$2M\mathbb{Z}$	0	$T\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0
R_{-+}	BDI, CII	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}_2$	0
R_{+-}	DIII, CI	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0
R_{+-}	BDI	$M\mathbb{Z}\oplus\mathbb{Z}$	0	0	0	$2M\mathbb{Z}\oplus 2\mathbb{Z}$	0	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$
R_{-+}	DIII	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}\oplus\mathbb{Z}$	0	0	0	$2M\mathbb{Z}\oplus 2\mathbb{Z}$	0
R_{+-}	CII	$2M\mathbb{Z}\oplus 2\mathbb{Z}$	0	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}\oplus\mathbb{Z}$	0	0	0
R_{-+}	CI	0	0	$2M\mathbb{Z}\oplus 2\mathbb{Z}$	0	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}_2\oplus\mathbb{Z}_2$	$M\mathbb{Z}\oplus\mathbb{Z}$	0

SnTe

Chiu & Schnyder 2014; Chiu, Yao, Ryu, PRB 2013; Morimoto & Furusaki PRB 2013; Shiozaki & Sato 2014

Gapless Topological Materials

Gapless topological materials

How about topology of gapless systems?

Consider e.g., metallic systems with FS or nodal superconductors

▶ Problem: Global topological number ill-defined (no gap!)

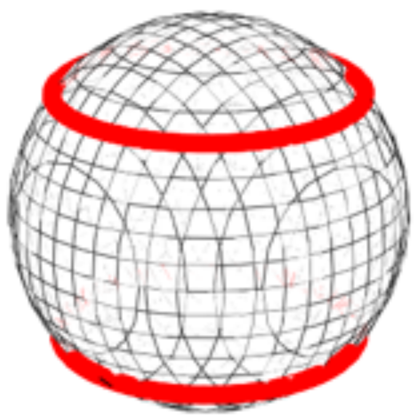
Solution: (assume translational symmetry)

⇒ Define momentum-dependent topological number

Example 1: nodal superconductor $\nu_{\mathcal{C}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \mathcal{F}(\mathbf{k}) dk_l$

where $\mathcal{C}$ interlinks with nodal line

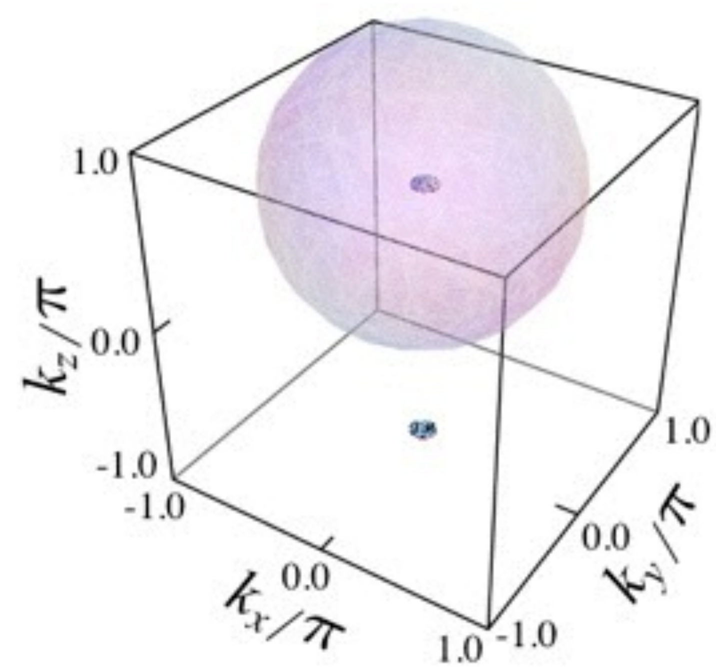
nodal lines on Fermi surface



Example 2: Weyl semi-metal

$$N_{S^2} = \frac{1}{2\pi} \int_{S^2} d^2\boldsymbol{k}\, \mathrm{Tr}\,[\mathcal{F}] \qquad \mathcal{F} = \nabla \times \mathcal{A}$$

where S^2 encloses nodal point

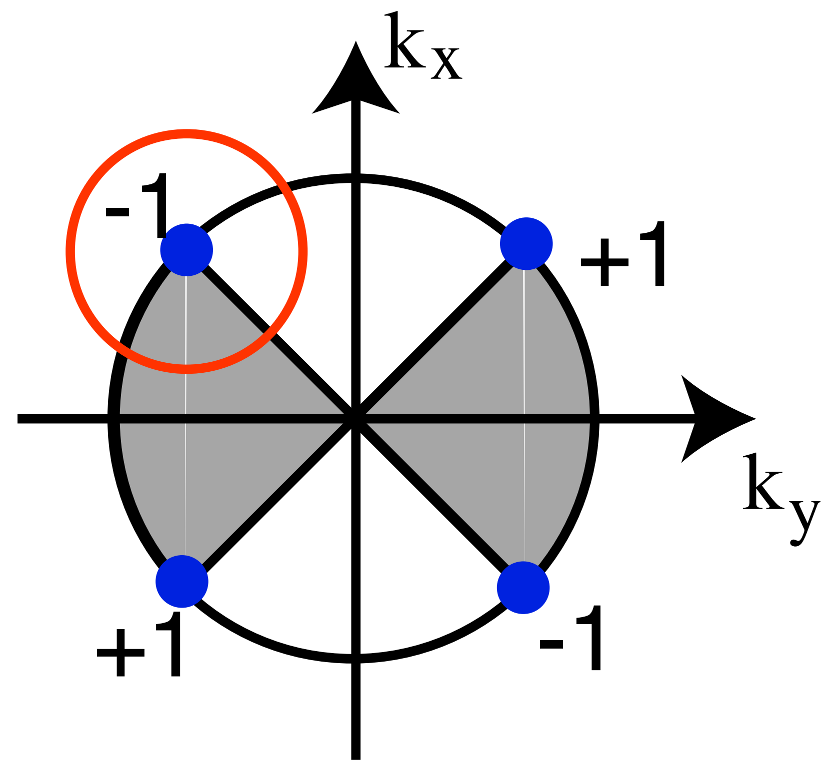


▶ Topological characteristics depend on the symmetries of Hamiltonian restricted to contour

Topologically stable point nodes in d_{x2-y2} -wave SCs

- Consider $d_{x^2-y^2}$ -wave superconductor

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} +\varepsilon_{\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta_{\boldsymbol{k}} & -\varepsilon_{\boldsymbol{k}} \end{pmatrix} \qquad \Delta_{\boldsymbol{k}} = \Delta_0(\cos k_x - \cos k_y)$$



Satisfies time-reversal symmetry *T* and particle-hole symmetry *C*

Combination of particle-hole symmetry and time-reversal symmetry gives

$$S\mathcal{H}(\boldsymbol{k})S^{\dagger} = -\mathcal{H}(\boldsymbol{k}) \qquad \text{with} \qquad S = TC = \sigma_2$$

In basis in which S is diagonal $\mathcal{H}(\boldsymbol{k})$ takes off-diagonal form: $\tilde{\mathcal{H}}(\boldsymbol{k}) = \begin{pmatrix} 0 & \varepsilon_{\boldsymbol{k}} - i\Delta_{\boldsymbol{k}} \\ \varepsilon_{\boldsymbol{k}} + i\Delta_{\boldsymbol{k}} & 0 \end{pmatrix}$

Spectrum flattening: $q(\boldsymbol{k}) = \dfrac{\varepsilon_{\boldsymbol{k}} + i\Delta_{\boldsymbol{k}}}{\sqrt{\varepsilon_{\boldsymbol{k}}^2 + \Delta_{\boldsymbol{k}}^2}}$ Consider: $q(\boldsymbol{k}) : S^1 \longrightarrow S^1$

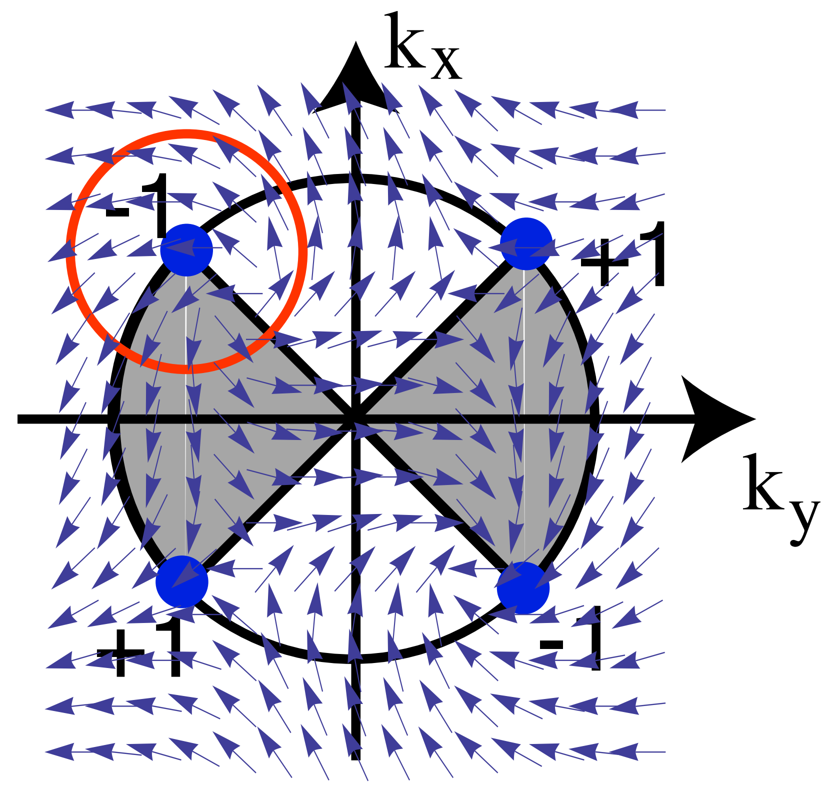
$\pi_1(S^1) = \mathbb{Z}$ $\Longrightarrow$ nodal points are protected by one-dimensional *winding number:*

$$\boxed{W_{\mathcal{L}} = \frac{1}{2\pi i} \oint_{\mathcal{L}} dk_l \, \mathrm{Tr}\left[q^{-1} \partial_{k_l} q\right] = \pm 1}$$

Note: $W_{\mathcal{L}}$ is invariant under path deformation.

Topologically stable point nodes in d_{x2-y2} -wave SCs

- Consider $d_{x^2-y^2}$ -wave superconductor



$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} +\varepsilon_{\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta_{\boldsymbol{k}} & -\varepsilon_{\boldsymbol{k}} \end{pmatrix} \qquad \Delta_{\boldsymbol{k}} = \Delta_0(\cos k_x - \cos k_y)$$

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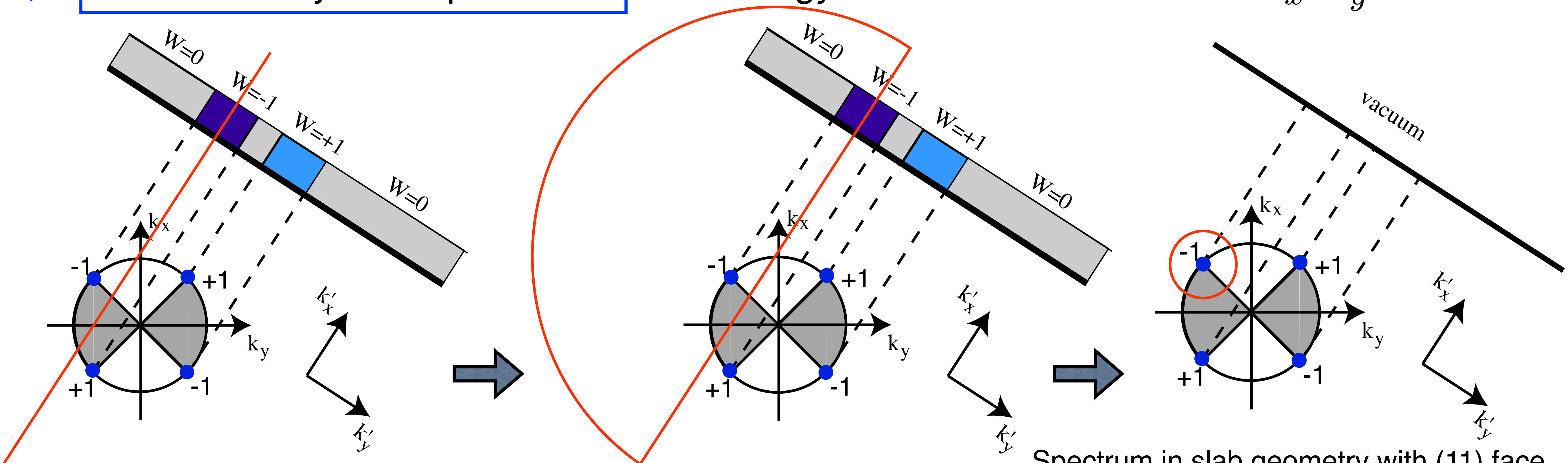
Note: $W_{\mathcal{L}}$ is invariant under path deformation.

Topologically stable point nodes in $d_{x^2-y^2}$ -wave SCs

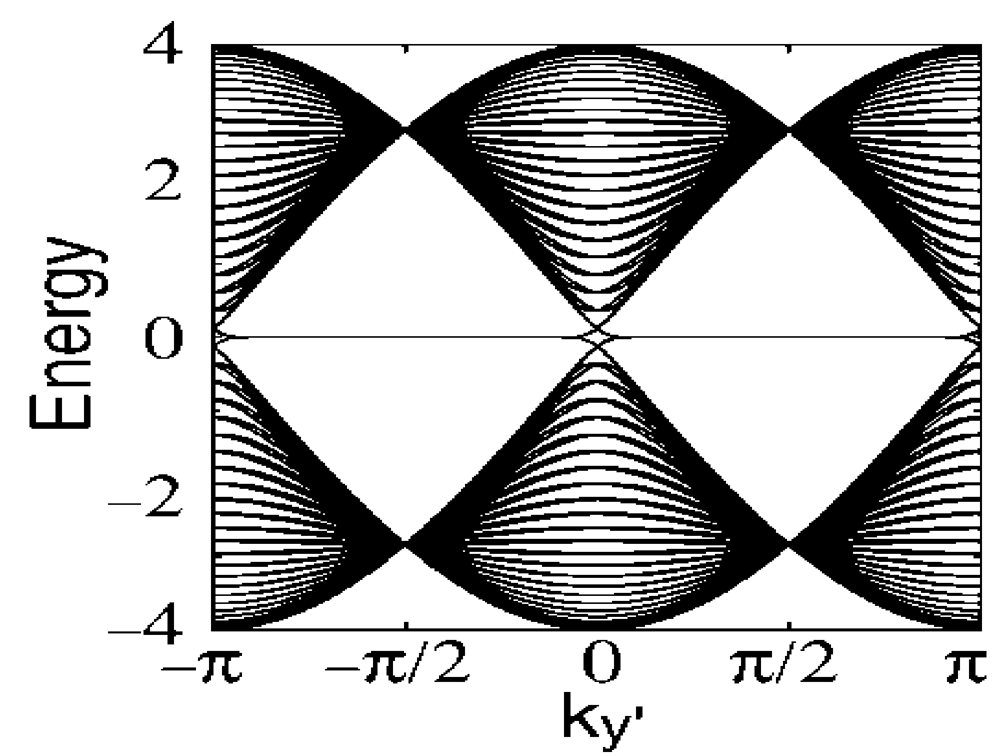
- Bulk-boundary correspondence: *zero-energy flat* bands on surface of $d_{x^2-y^2}$ SC



$$W_{\mathcal{L}} = \frac{1}{2\pi i} \oint_{\mathcal{L}} dk_l \, \mathrm{Tr}\left[q^{-1} \partial_{k_l} q \right] = \pm 1$$

- Zero-energy surface states on all edges except the (10) and (01) face.

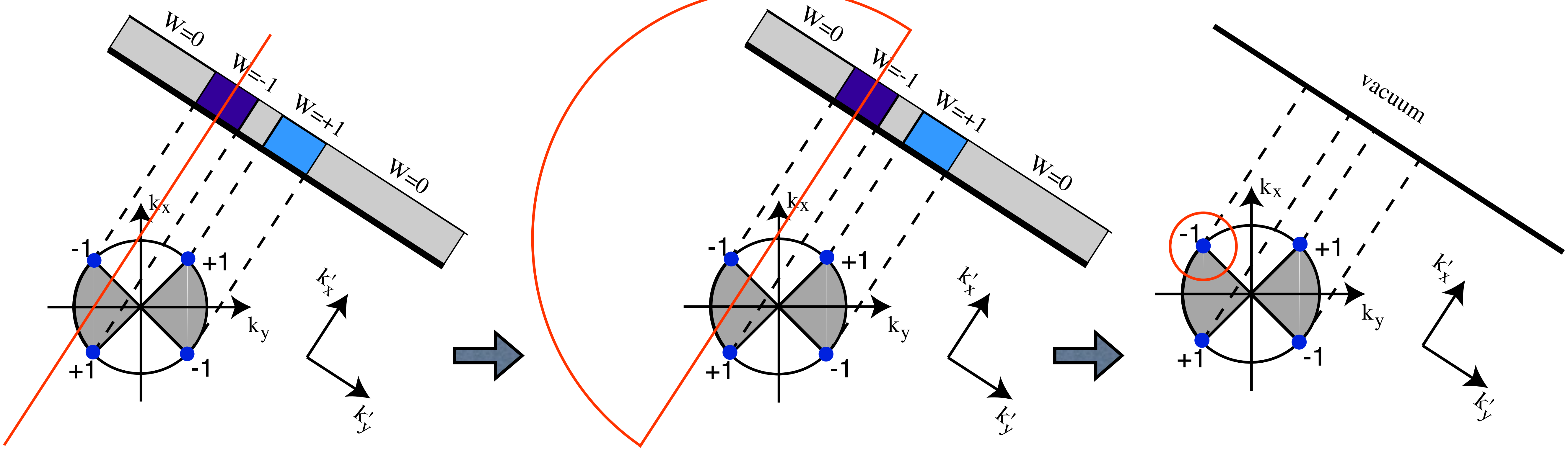
Spectrum in slab geometry with (11) face



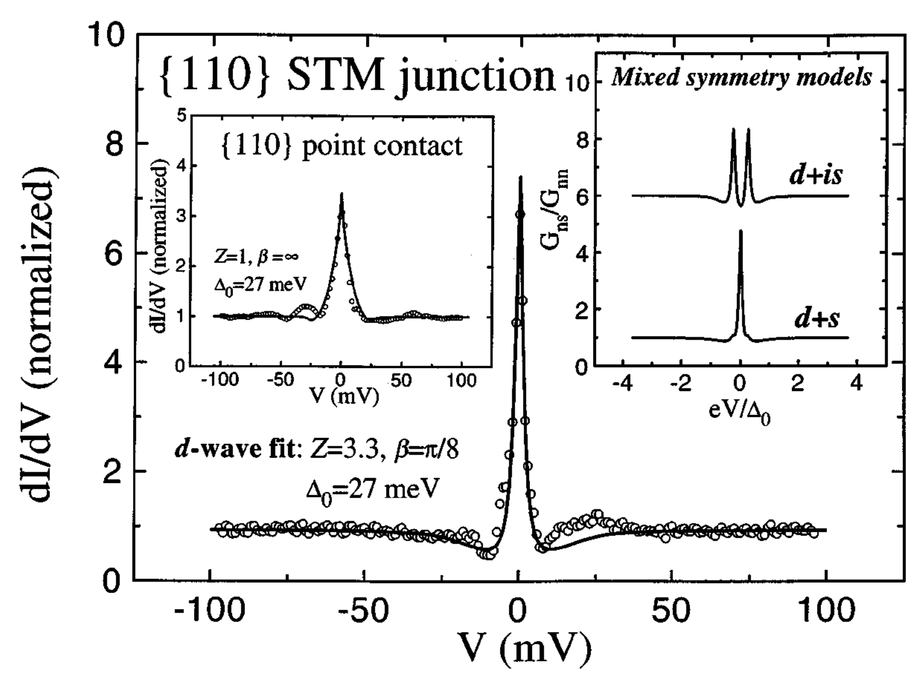
[cf, Hu, PRL 94, Wakabayashi et al. '05]

Topologically stable point nodes in $d_{x^2-y^2}$-wave SCs

- ▶ Bulk-boundary correspondence: *zero-energy flat* bands on surface of $d_{x^2-y^2}$ SC



- ▶ Experimental observation in high-Tc cuprates:



[Wei et al. PRL '98]

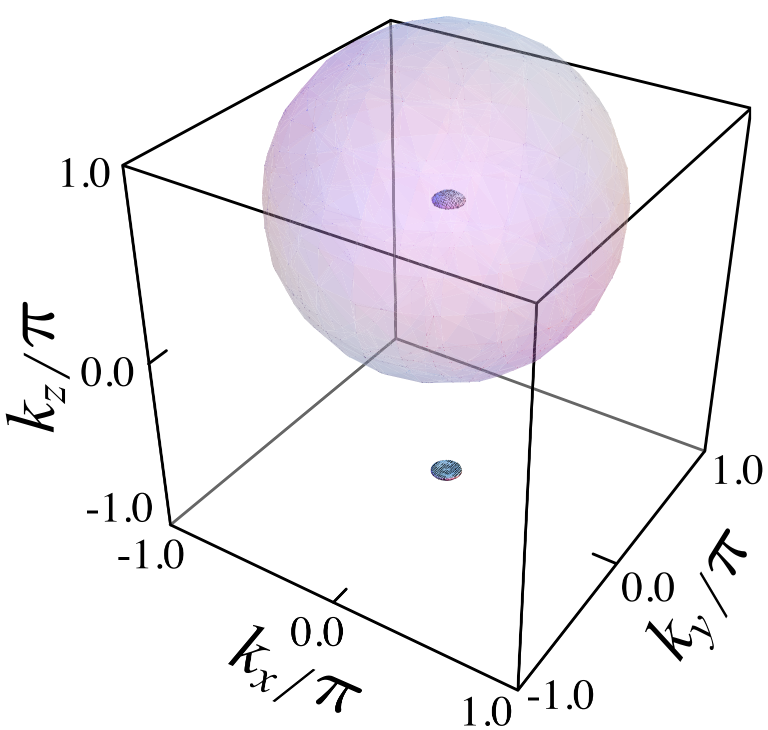
[Kashiwaya et al. '95, Alff et al. '97]

Zero energy surface modes in ^{3}He A phase

▶ Example II: ^{3}He A phase (equivalent to Weyl semi-metal)
(possibly realized in pyrochlore iridates, TI-multilayer, ferromagnetic SCs)

$$H(\mathbf{k}) = \begin{pmatrix} +\varepsilon(\mathbf{k}) & \Delta(\mathbf{k}) \\ \Delta^\dagger(\mathbf{k}) & -\varepsilon(\mathbf{k}) \end{pmatrix} \qquad \Delta(\mathbf{k}) = (\mathbf{d_k} \cdot \vec{\sigma})(i\sigma_2)$$

d-vector points along z-axis: $\mathbf{d_k} = \hat{z}\Delta_0(\sin k_x + i \sin k_y)$



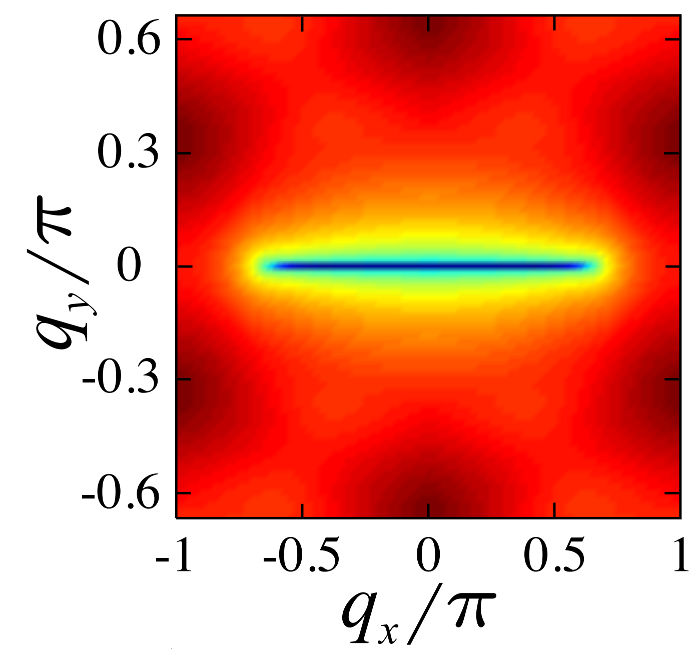
Topologically stable Fermi points protected by two-dimensional Chern number:

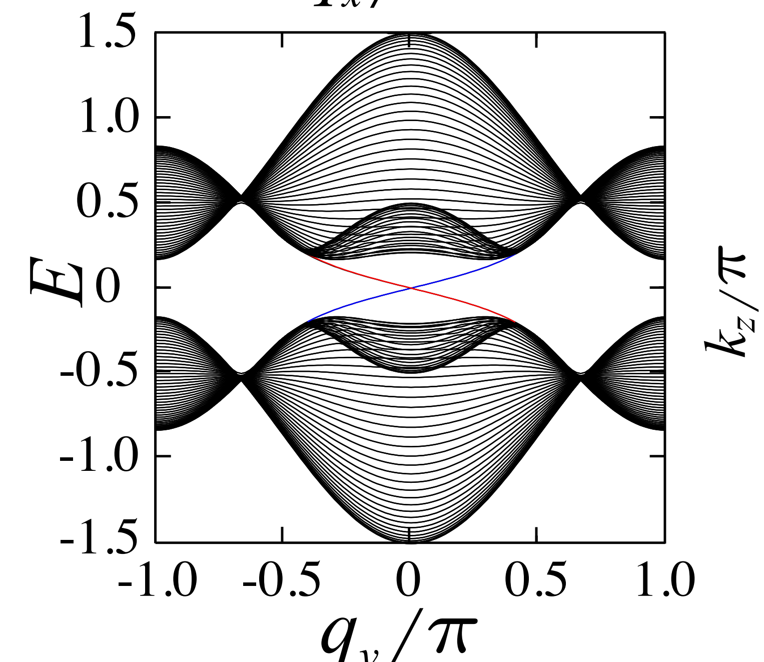
$$N_{\tilde{S}^2} = \frac{1}{8\pi}\int_{\tilde{S}^2} d^2\boldsymbol{k}\, \varepsilon^{\mu\nu}\boldsymbol{n}\cdot\left[\partial_{k_\mu}\boldsymbol{n} \times \partial_{k_\nu}\boldsymbol{n}\right]$$

with:

$$\boldsymbol{n} = \frac{1}{\sqrt{\varepsilon_{\boldsymbol{k}}^2 + |\Delta_{\boldsymbol{k}}|^2}} \begin{pmatrix} \varepsilon_{\boldsymbol{k}} \\ \mathrm{Re}\Delta_{\boldsymbol{k}} \\ \mathrm{Im}\Delta_{\boldsymbol{k}} \end{pmatrix}$$

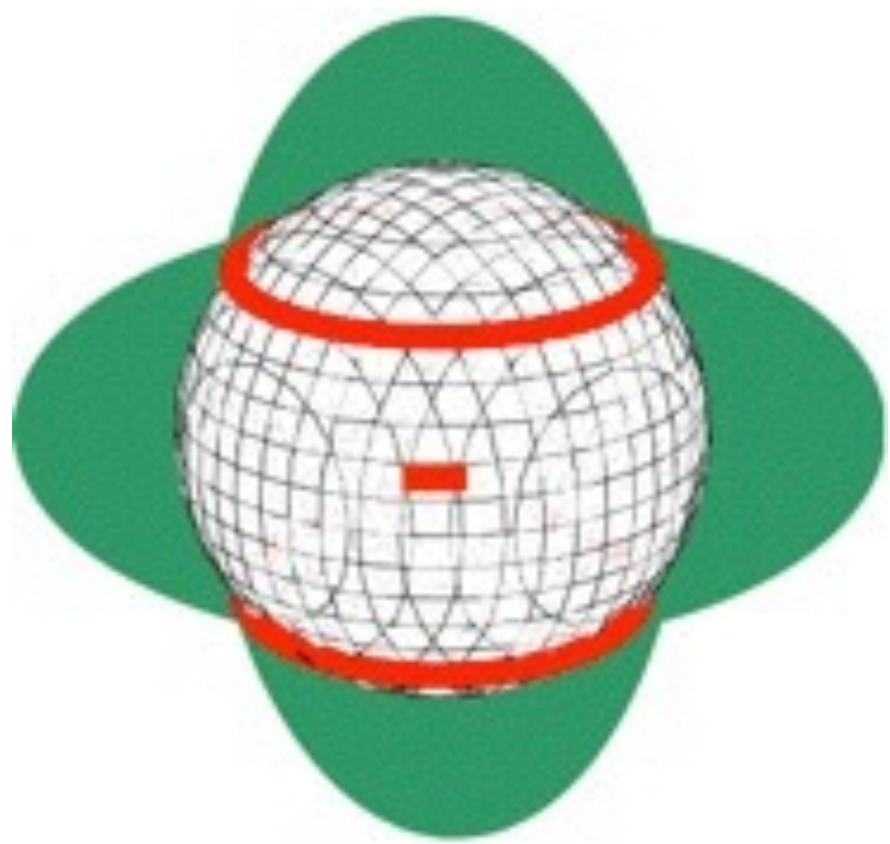
Surface spectrum in slab geometry with (111) face





▶ Bulk-boundary correspondence:
surface Fermi arc connecting the projected nodal points

Nodal non-centrosymmetric superconductors



Nodal topological superconductors

Consider nodal topological superconductor

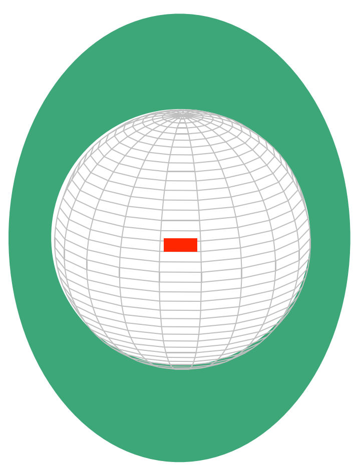
Non-centro SC: $\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon_{\mathbf{k}}\sigma_0 + \lambda\,\mathbf{g}_{\mathbf{k}}\cdot\vec{\sigma} & [\Delta_s\sigma_0 + \Delta_t\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}](i\sigma_y) \\ (-i\sigma_y)[\Delta_s\sigma_0 + \Delta_t\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}] & -\varepsilon_{\mathbf{k}}\sigma_0 - \lambda\,\mathbf{g}_{\mathbf{k}}\cdot\vec{\sigma}^* \end{pmatrix}$

Spin-split Fermi surfaces: $\xi^{\pm}_{\mathbf{k}} = \varepsilon_{\mathbf{k}} \pm \lambda|\mathbf{g}_{\mathbf{k}}|$

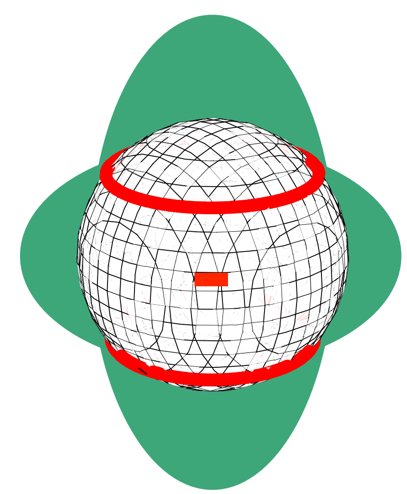
Gaps on the two Fermi surfaces: $\boxed{\Delta^{\pm}_{\boldsymbol{k}} = \Delta_s \pm \Delta_t\,|\boldsymbol{d}_{\boldsymbol{k}}|}$

$\Delta_s > \Delta_t$ | $\Delta_s \sim \Delta_t$ | $\Delta_s < \Delta_t$

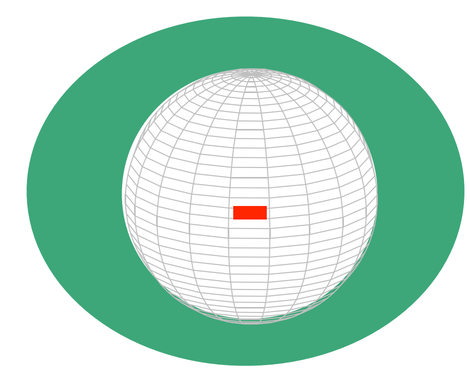
negative helicity Fermi surface



full gap



line nodes



full gap

$\Delta_s \sim \Delta_t \quad \Longrightarrow$ *nodal lines* on negative helicity FS

Topologically non-trivial

Nodal topological superconductors

Consider nodal topological superconductor

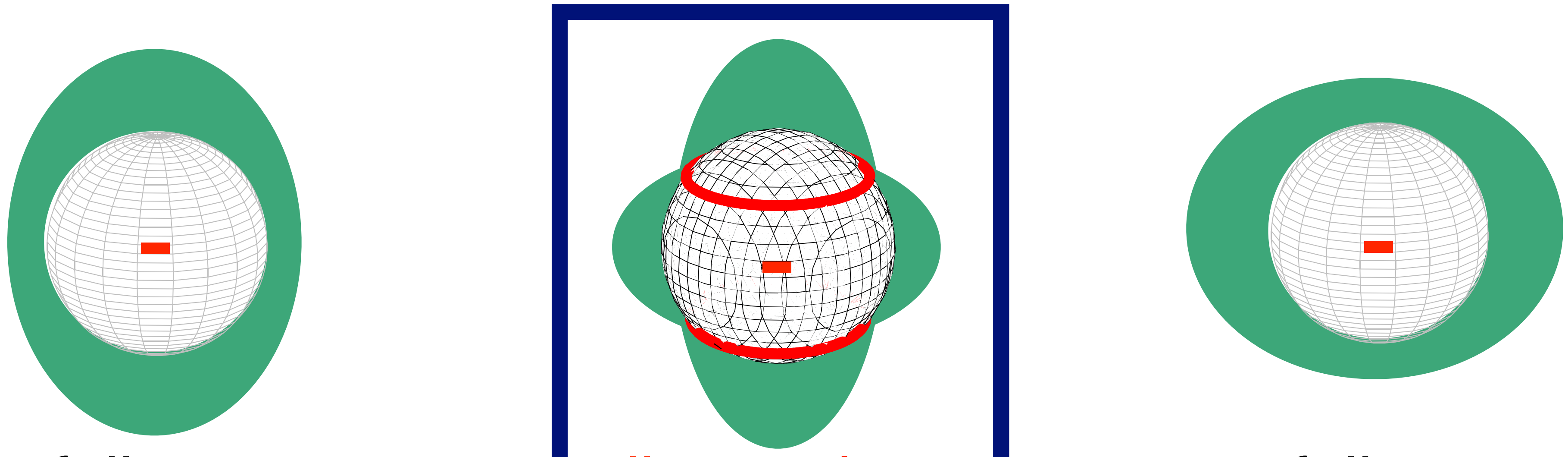
Non-centro SC: $\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon_{\mathbf{k}}\sigma_0 + \lambda\, \mathbf{g}_{\mathbf{k}} \cdot \vec{\sigma} & [\Delta_s \sigma_0 + \Delta_t \mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}](i\sigma_y) \\ (-i\sigma_y)[\Delta_s \sigma_0 + \Delta_t \mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}] & -\varepsilon_{\mathbf{k}}\sigma_0 - \lambda\, \mathbf{g}_{\mathbf{k}} \cdot \vec{\sigma}^* \end{pmatrix}$

Spin-split Fermi surfaces: $\xi^{\pm}_{\mathbf{k}} = \varepsilon_{\mathbf{k}} \pm \lambda |\mathbf{g}_{\mathbf{k}}|$

Gaps on the two Fermi surfaces: $\boxed{\Delta^{\pm}_{\boldsymbol{k}} = \Delta_s \pm \Delta_t \, |\boldsymbol{d}_{\boldsymbol{k}}|}$

$\Delta_s > \Delta_t$ $\Delta_s \sim \Delta_t$ $\Delta_s < \Delta_t$

negative helicity Fermi surface



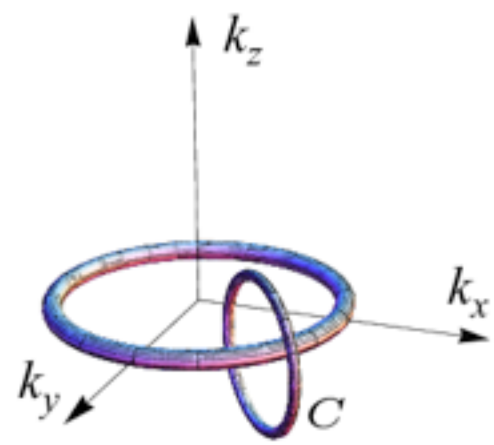
full gap *line nodes* *full gap*

$\Delta_s \sim \Delta_t \quad \Longrightarrow$ *nodal lines* on negative helicity FS

Topologically non-trivial

Nodal topological superconductors

▶ Topological characteristics depend on the symmetries of BdG Hamiltonian restricted to contour $\mathcal{C}$.



Classification

Symmetry				dim		
Class	T	P	S	1	2	3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	1	-1	1	0	0	$\mathbb{Z}$

$$\mathbf{d}_{\mathbf{k}} = (\sin k_x + \sin k_y, \sin k_x + \sin k_y, \sin k_z)^{\mathrm{T}}$$

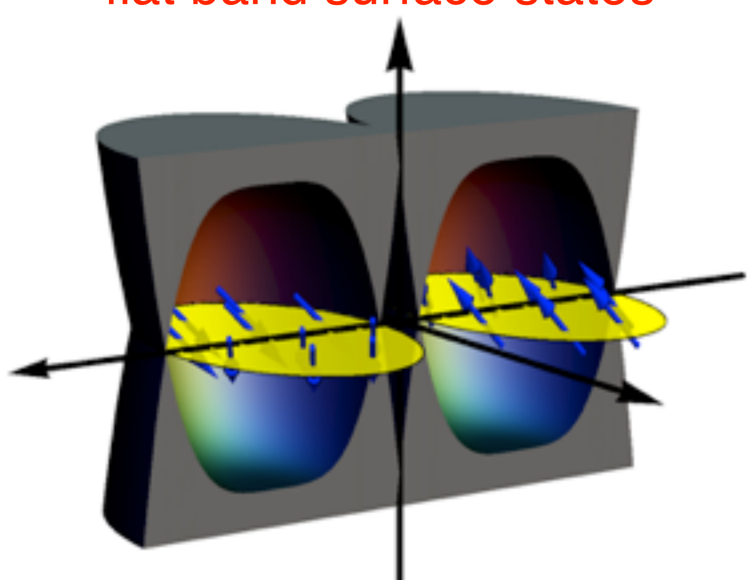
$$\Delta_s \sim \Delta_t$$

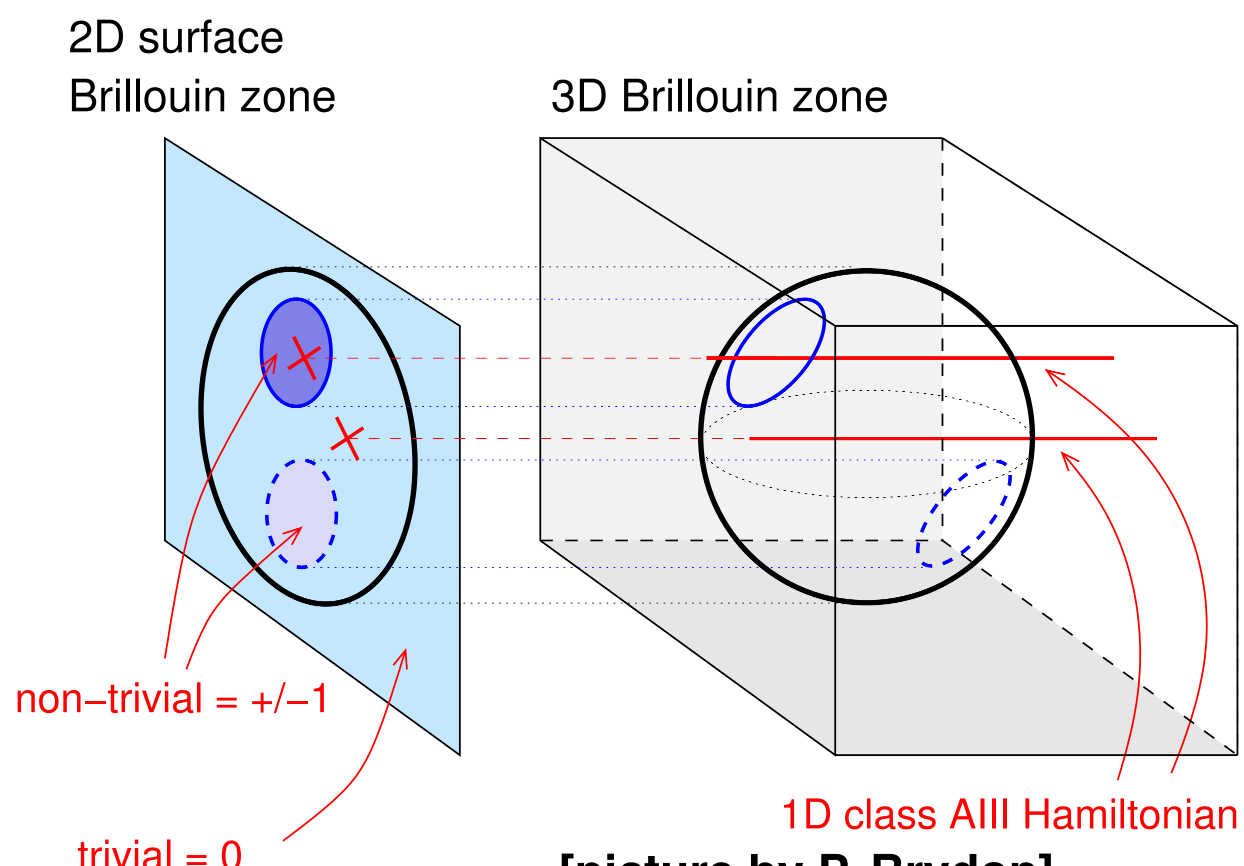
(i) 1D contour *is not* centrosymmetric: TRS ✗ PHS ✗ TRS+PHS (chiral sym S) ✓

AIII: 1D Winding number:

$$W_C = \frac{1}{2\pi} \oint_C dk_l \, \partial_{k_l} \left[\arg(\xi^-_{\mathbf{k}} + i\Delta^-_{\mathbf{k}}) \right]$$

flat band surface states





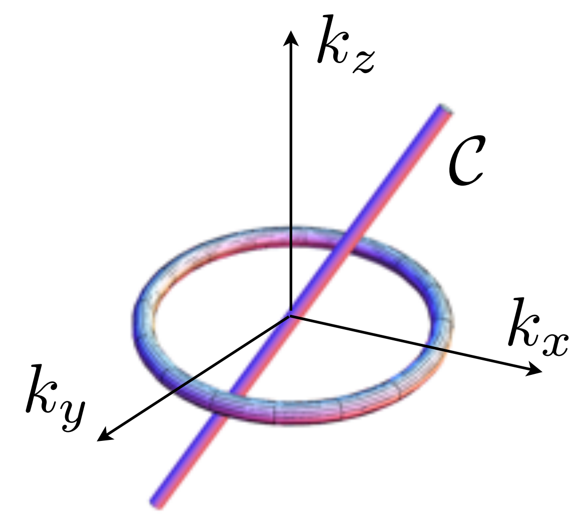
[picture by P. Brydon]

Nodal topological superconductors

- Topological characteristics depend on the symmetries of BdG Hamiltonian restricted to contour $\mathcal{C}$.

$$\mathbf{d}_{\mathbf{k}} = (\sin k_x, \sin k_y, \sin k_z)^{\mathrm{T}}$$

$$\Delta_s \sim \Delta_t$$



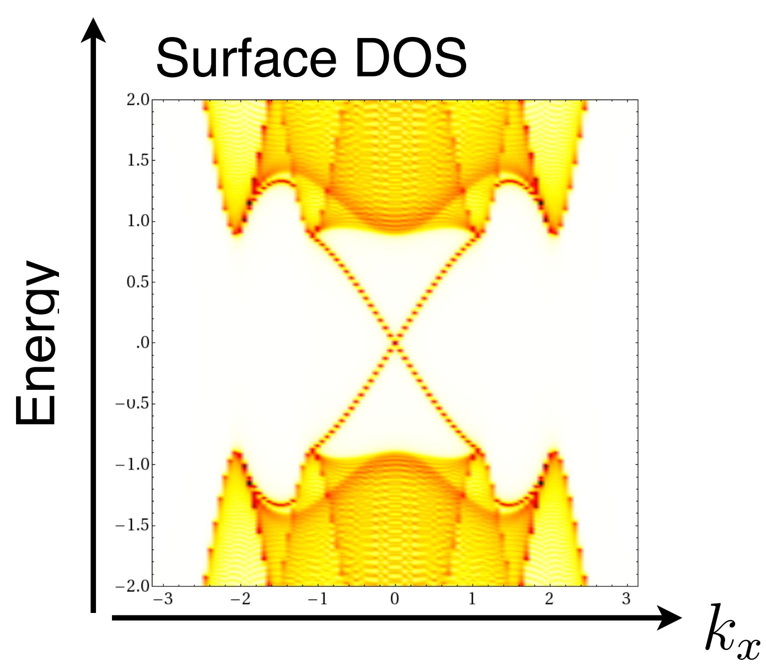
Classification

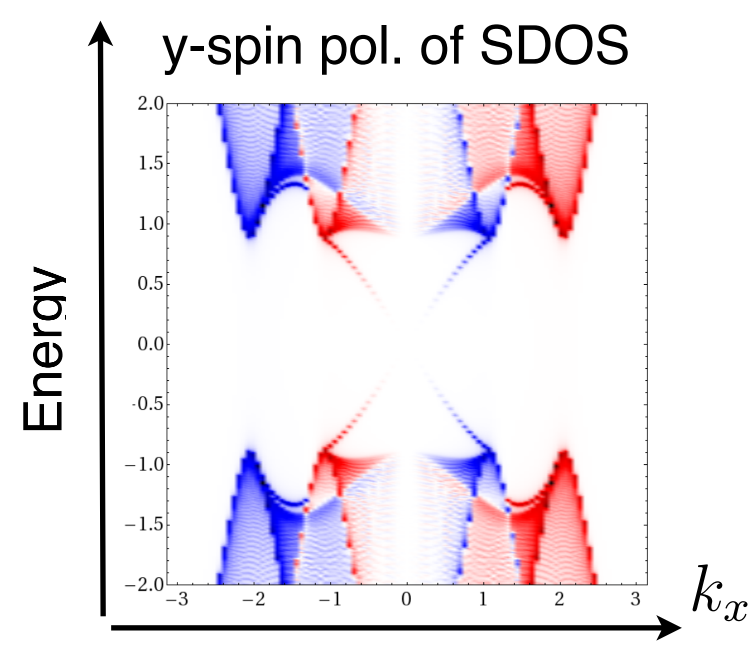
Class	T	P	S	dim 1	dim 2	dim 3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	1	-1	1	0	0	$\mathbb{Z}$

(ii) 1D contour *is* centrosymmetric: TRS ✓ PHS ✓ TRS+PHS (chiral sym S) ✓

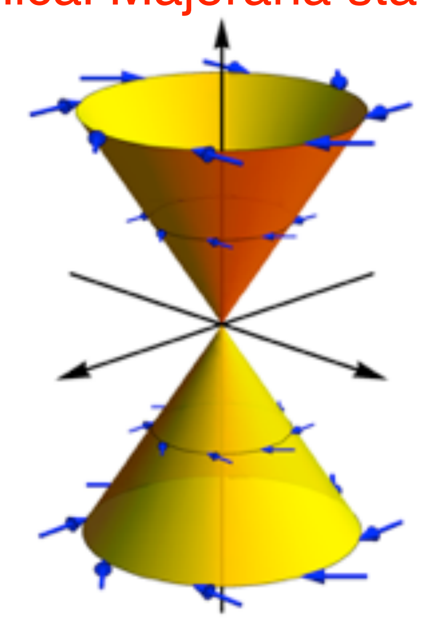
DIII: 1D Z_2 number:

$$N_{\mathcal{C}}^{1\mathrm{D}} = \prod_{a=1}^{2} \frac{\mathrm{Pf}\left[\omega(\Lambda_a)\right]}{\sqrt{\det\left[\omega(\Lambda_a)\right]}} = \prod_{a=1}^{2} \frac{\mathrm{Pf}\left[q^T(\Lambda_a)\right]}{\sqrt{\det\left[q(\Lambda_a)\right]}} = \pm 1$$





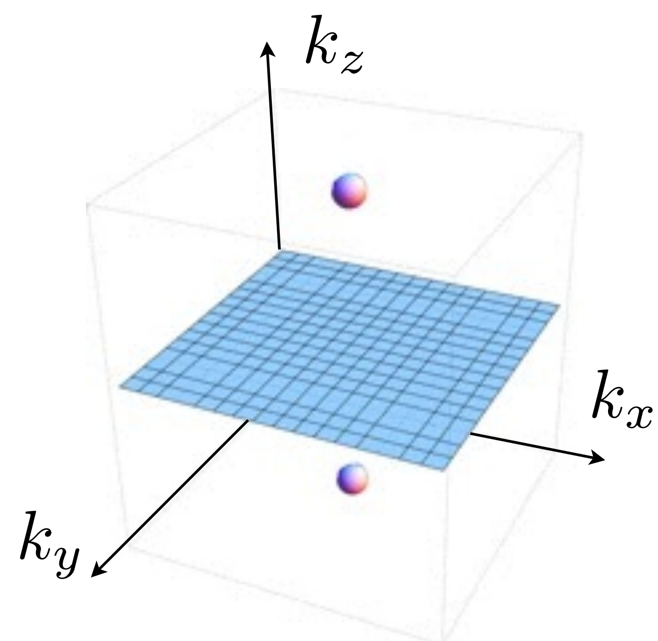
helical Majorana state



Nodal topological superconductors

▶ Topological characteristics depend on the symmetries of BdG Hamiltonian restricted to contour $\mathcal{C}$.

$$\mathbf{d}_{\mathbf{k}} = (\sin k_y, -\sin k_x, 0)^{\mathrm{T}}$$
$$\Delta_t \neq 0, \quad \Delta_s = 0$$



Classification

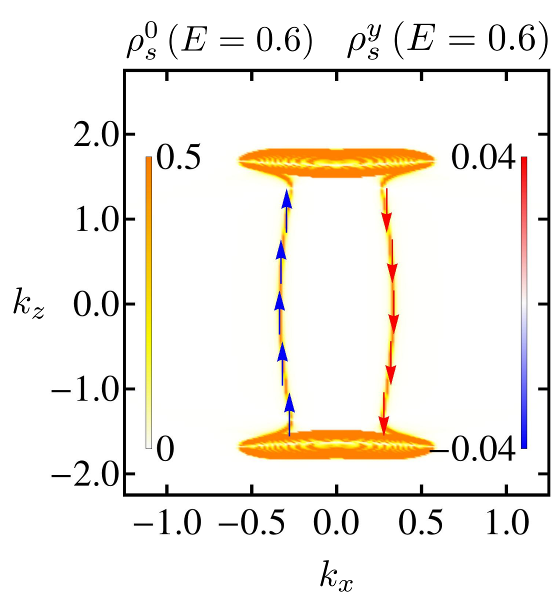
	Symmetry			dim		
Class	T	P	S	1	2	3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	1	-1	1	0	0	$\mathbb{Z}$

(iii) 2D contour *is* centrosymmetric: TRS ✓ PHS ✓ TRS+PHS (chiral sym S) ✓

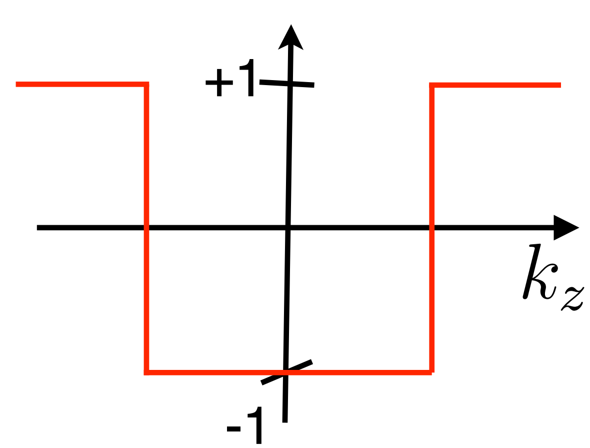
DIII: 2D Z_2 number:

$$N_E^{\mathrm{1D}} = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[\omega(\Lambda_a)\right]}{\sqrt{\det\left[\omega(\Lambda_a)\right]}} = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[q^T(\Lambda_a)\right]}{\sqrt{\det\left[q(\Lambda_a)\right]}} = \pm 1$$

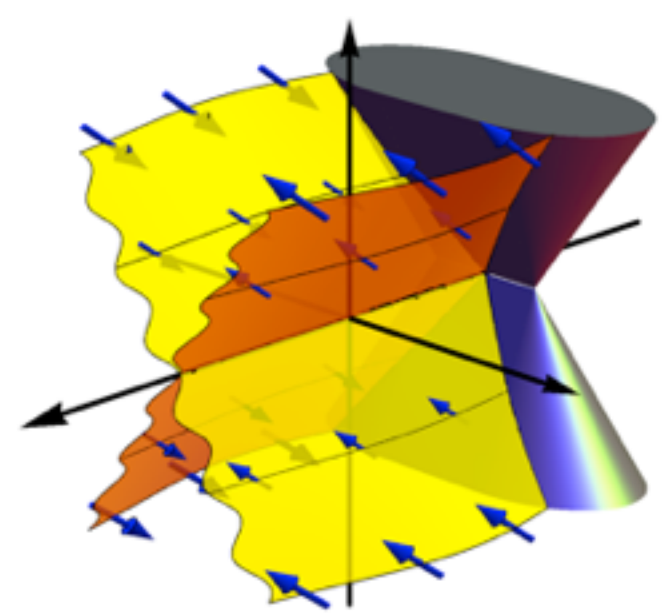
Surface density of states



2D Z_2 number



arc surface state



Topological classification of gapless materials

Classification for Fermi surfaces off high-symmetry points

Two-dimensional systems

Class	T	P	S	line	point
A	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\boxed{\mathbb{Z}}$
AI	+1	0	0	$\mathbb{Z}$	0
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
DIII	-1	+1	1	0	$\mathbb{Z}_2$
AII	-1	0	0	$\mathbb{Z}$	0
CII	-1	-1	1	0	$\mathbb{Z}$
C	0	-1	0	0	0
CI	+1	-1	1	0	0

d-wave SC

Three-dimensional systems

Class	surface	line	point
A	$\mathbb{Z}$	0	$\boxed{\mathbb{Z}}$
AIII	0	$\boxed{\mathbb{Z}}$	0
AI	$\mathbb{Z}$	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}_2}$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CII	0	$\mathbb{Z}$	0
C	0	0	$\mathbb{Z}$
CI	0	0	0

^{3}He A phase / Weyl semimetal

TRI SC w/ S^z-spin conserv.

Non-centro SC

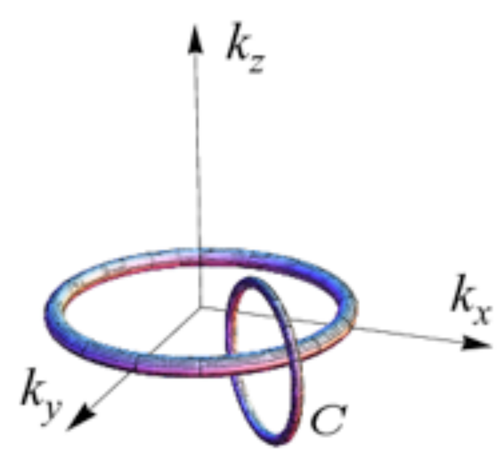
(NB: $\mathbb{Z}_2$ invariant only protects surface states, but not bulk nodes!)

$\mathbb{Z}$: integer classification
$\mathbb{Z}_2$: binary classification
0 : no top. stable nodes

- Topological invariants:

$$\nu_{\mathcal{C}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \mathcal{F}(\mathbf{k}) dk_l$$

- Classification depends on whether contour is centro-symmetric or not (i.e. invariant under k-> -k)



[Matsuura et al. NJP 15, 065001 (2013)]
[Zhao, Wang, PRL (2013)]

Acknowledgments

Theory

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Experiment

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