Topological insulators and superconductors

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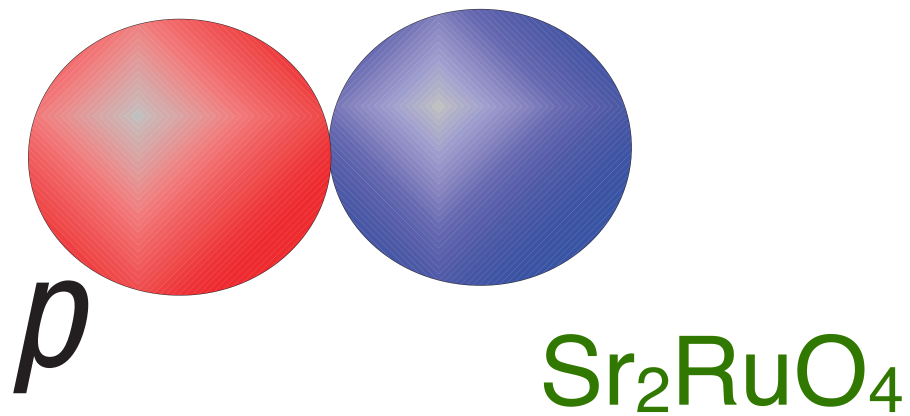


Topological superconductors

- Topological superconductors in 1D: Kitaev model
- Topological superconductors in 2D: chiral p-wave SC
- Helical superconductors (top. SCs w/ TRS)
- 3D time-reversal invariant superconductor

Bogoliubov-de Gennes theory for superconductors

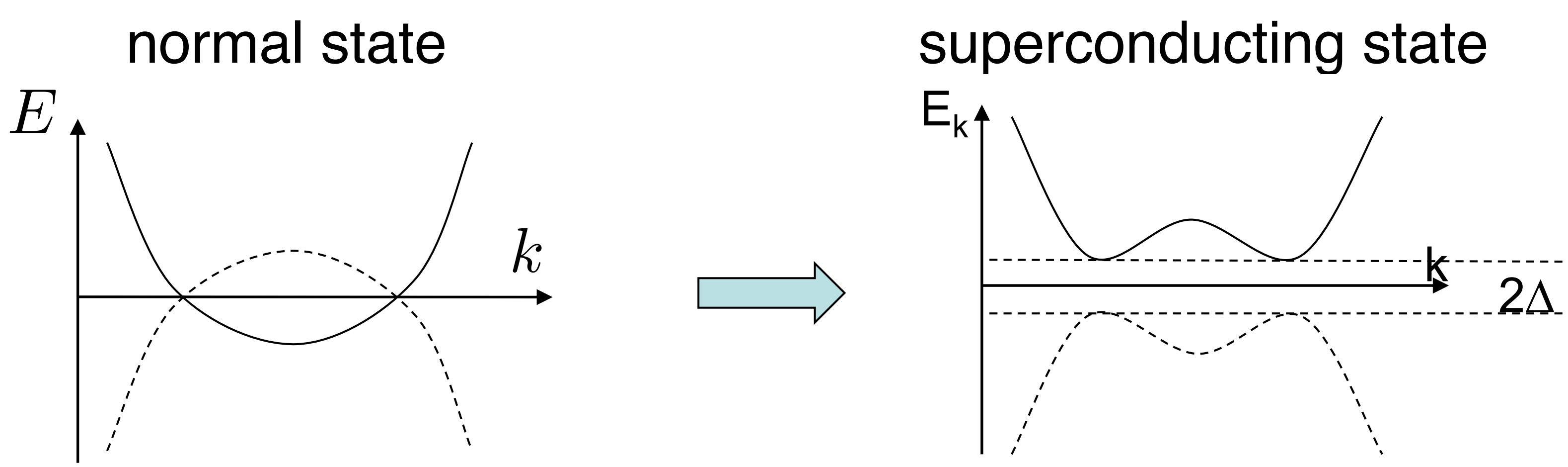
Superconductor = Cooper pairs (boson) + Bogoliubov quasiparticles (fermions)

BCS mean field theory: $c^\dagger c \, c^\dagger c \quad \Rightarrow \quad \langle c^\dagger c^\dagger \rangle c\, c = \Delta^* c\, c$

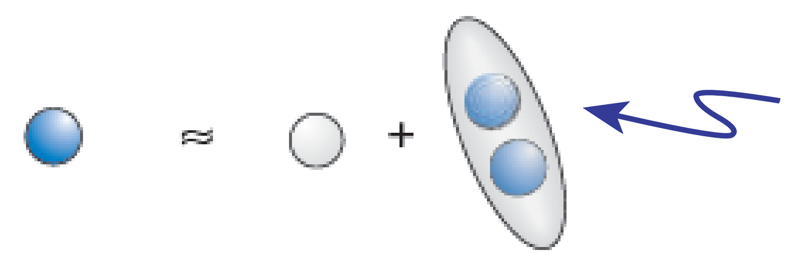
$$H = \frac{1}{2} \sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger & c \end{pmatrix} H_{\mathrm{BdG}} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$$

Bogoliubov-de Gennes Hamiltonian

$$H_{\mathrm{BdG}} = \begin{pmatrix} h_0 & \Delta \\ \Delta^\dagger & -h_0^T \end{pmatrix}$$



Built-in anti-unitary particle-hole symmetry:



Cooper pair

$$\boxed{C \mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) C^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})} \qquad C = \tau_x \mathcal{K} \qquad C^2 = +1 \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(for triplet pairing)

Particle-hole symmetry: $C\psi_{+\mathbf{k},+E} = \tau_x \psi^*_{-\mathbf{k},-E} \quad \Rightarrow \quad \gamma^\dagger_{\boldsymbol{k},E} = \gamma_{-\boldsymbol{k},-E}$

Bogoliubov-de Gennes theory for superconductors

Superconductor = Cooper pairs (boson) + Bogoliubov quasiparticles (fermions)

BCS mean field theory: $\quad c^\dagger c\, c^\dagger c \quad \Rightarrow \quad \langle c^\dagger c^\dagger \rangle c\, c = \Delta^* c\, c$

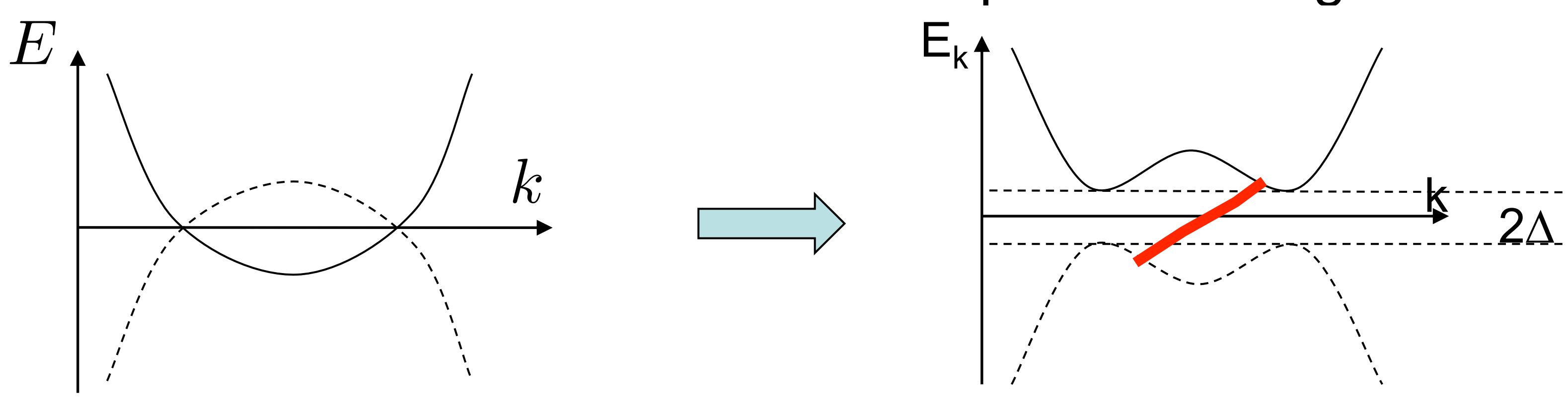
$$H = \frac{1}{2} \sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger & c \end{pmatrix} H_{\mathrm{BdG}} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$$

Bogoliubov-de Gennes Hamiltonian

$$H_{\mathrm{BdG}} = \begin{pmatrix} h_0 & \Delta \\ \Delta^\dagger & -h_0^T \end{pmatrix}$$

normal state

superconducting state



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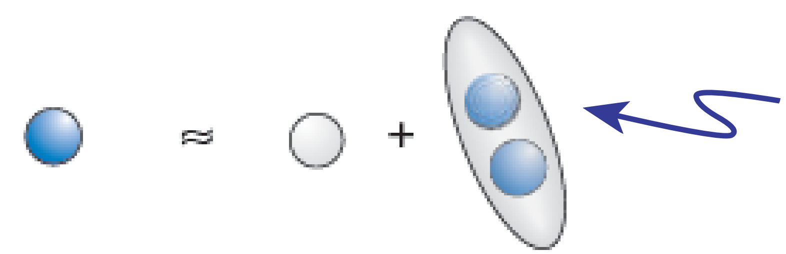

Cooper pair

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(for triplet pairing)

Particle-hole symmetry + bulk-boundary correspondence:

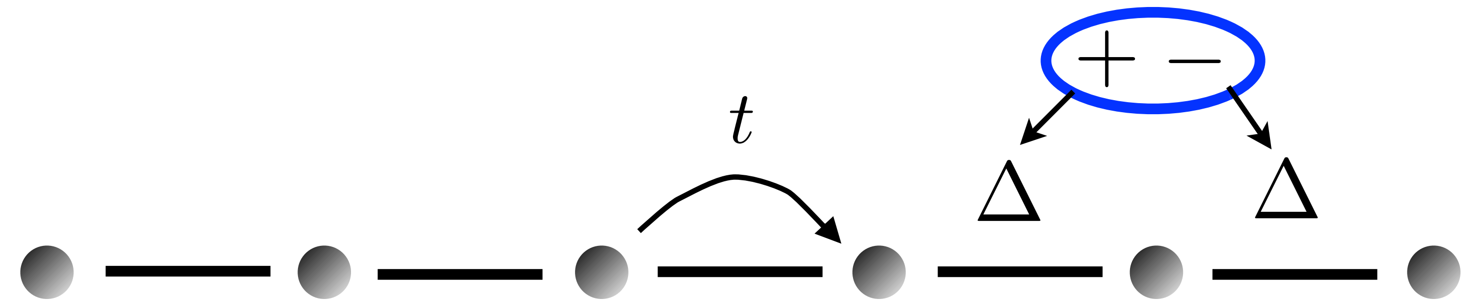
Majorana edge state at zero energy

1D topological superconductor: Majorana chain

[Kitaev 2000]

One-dimensional spinless p-wave superconductor: Majorana chain

Experiments: InSb-nanowire-heterostructures



Hamiltonian:

$$\mathcal{H} = \sum_j \left[t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j - \mu c_j^\dagger c_j + \Delta(c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1}) \right]$$

in momentum space:

$$\mathcal{H} = \frac{1}{2} \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \mathcal{H}_{\mathrm{BdG}}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

$$\mathcal{H}_{\mathrm{BdG}}(k) = \mathbf{d}(k) \cdot \vec{\tau}$$

$$d_x(k) = 2i\Delta \sin k \quad d_y(k) = 0$$

$$d_z(k) = 2t \cos k - \mu$$

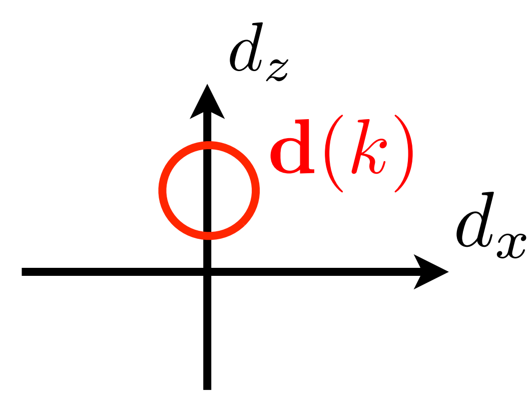
Particle-hole symmetry:

$$\tau_x \mathcal{H}_{\mathrm{BdG}}^*(k) \tau_x = -\mathcal{H}_{\mathrm{BdG}}(-k)$$

Time-reversal symmetry:

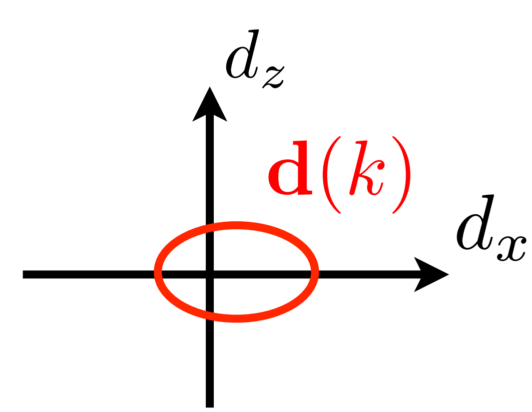
$$\tau_z \mathcal{H}_{\mathrm{BdG}}^*(k) \tau_z = +\mathcal{H}_{\mathrm{BdG}}(-k)$$

energy spectrum: $E_\pm = \pm |\mathbf{d}(k)|$



$|\mu| > 2t$:

trivial super-conductor



$|\mu| < 2t$:

topological superconductor

1D topological superconductor: Majorana chain

[Kitaev 2000]

To reveal zero-energy edge states, consider different viewpoint: **Majorana representation**

> Majorana fermion:
> Particle = Antiparticle

$$c_j = \frac{1}{2}\left(\gamma_{1j} + i\gamma_{2j}\right) \qquad c_j^\dagger = \frac{1}{2}\left(\gamma_{1j} - i\gamma_{2j}\right)$$

Anti-commutation relations: $\left\{\gamma_{lj}, \gamma_{l'j'}\right\} = 2\delta_{ll'}\delta_{jj'} \qquad \left(\gamma_{lj}\right)^2 = 1$

$\Longrightarrow$ Majorana chain for spinless fermions

$$H = \frac{i}{2}\sum_j \left[-\mu\gamma_{1j}\gamma_{2j} + (\Delta - t)\gamma_{2j}\gamma_{1j+1} + (\Delta + t)\gamma_{1j}\gamma_{2j+1}\right]$$

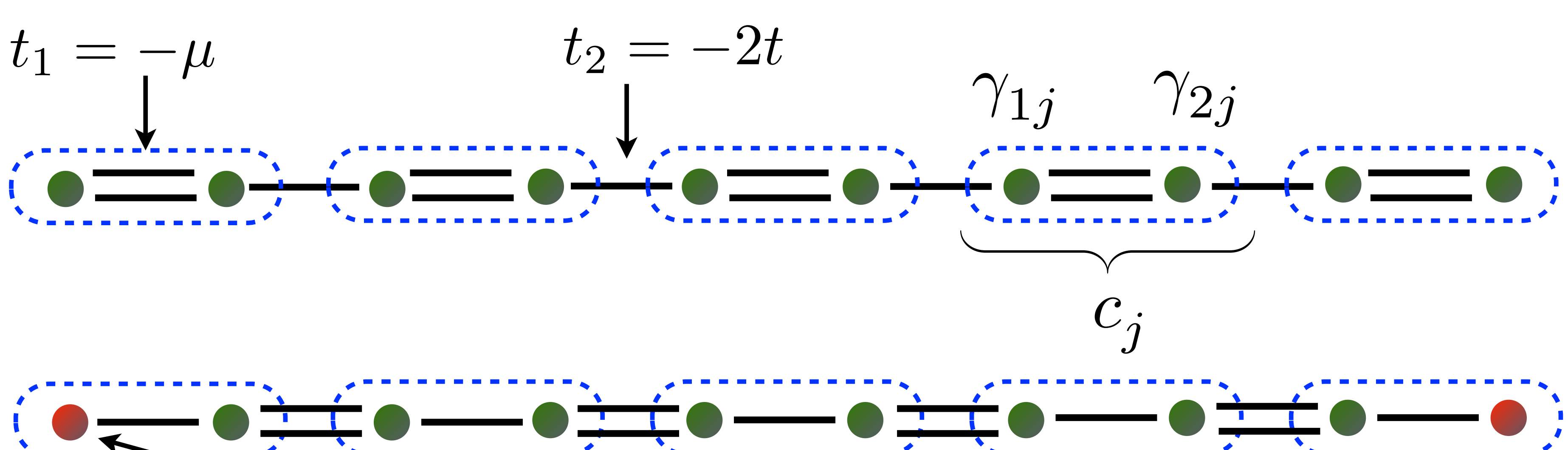
for $\Delta = -t$: nearest neighbor Majorana chain

$t_1 > t_2$:

trivial superconductor

$t_1 < t_2$:

topological superconductor



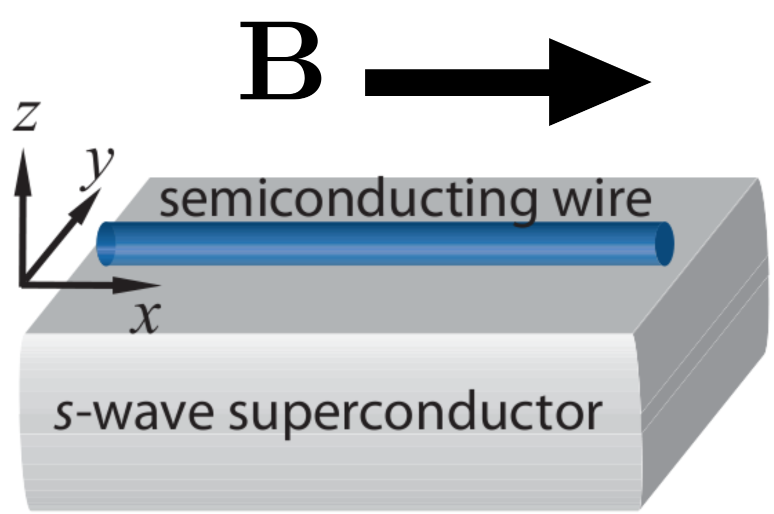
— zero-energy Majorana end states

Majorana end state

Experimental detection of 1D spinless topological SC

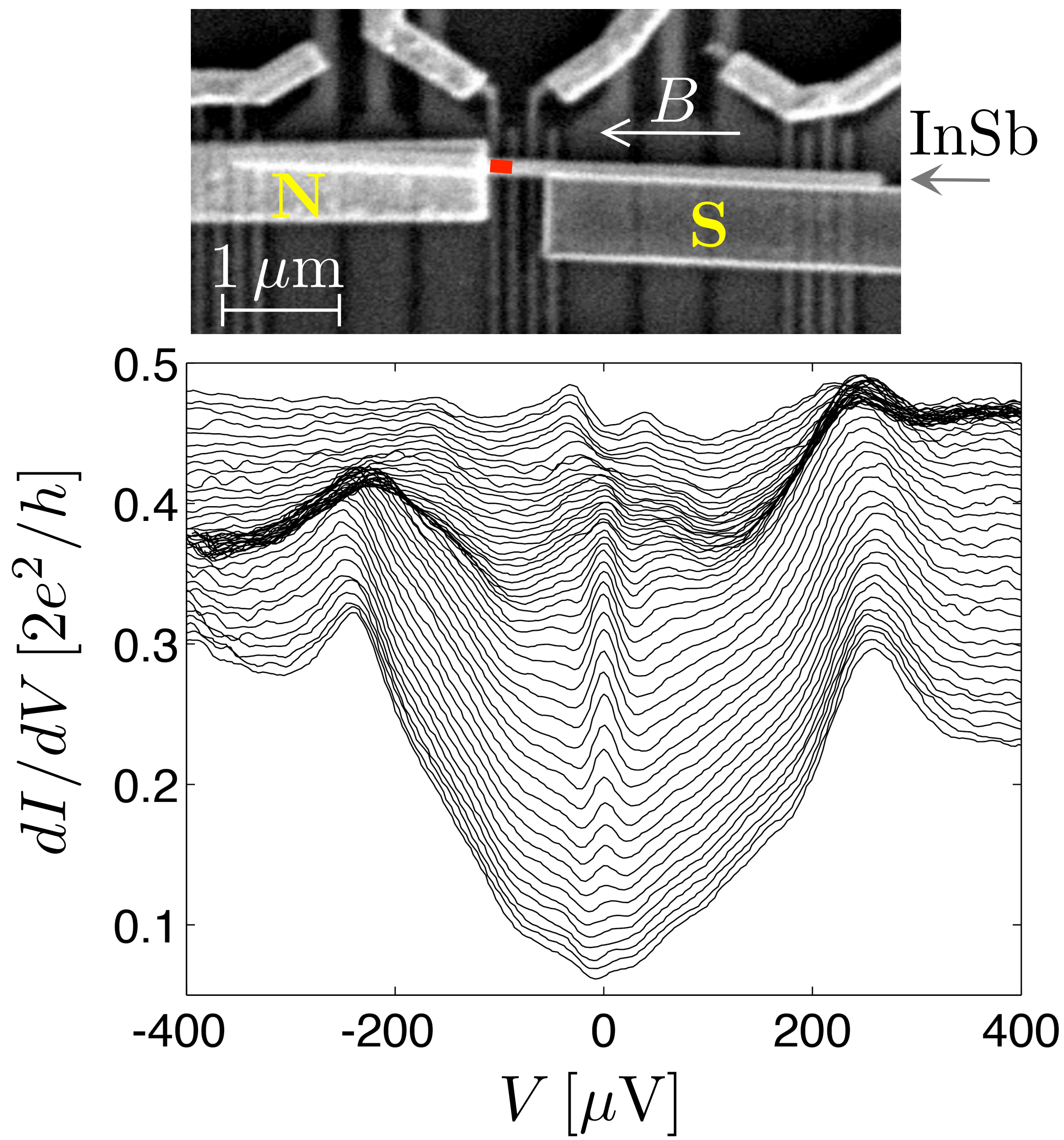
1D spinless chiral p-wave superconductor is likely (?) realized in InSb-nanowire-heterostructures

magnetic field B



- Condition for topological phase:

$$B \propto E_{\text{Zeeman}} > \sqrt{\Delta^2 - \mu^2}$$



Differential tunneling conductance as a function of magnetic field B

[Sau, Lutchyn, Tewari, das Sarma, et al 2009]
[Oreg, von Oppen, et al 2010]

[after Alicea, Rep. Prog. Phys. 2012]

[Mourik, Kouwenhoven et al, Science 2012]

Two-dimensional spinless chiral p-wave SC

[Read & Green 00]

Lattice BdG model:
(on square lattice)

$$H_{\rm BdG} = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}} & c_{-\mathbf{k}} \end{pmatrix} \mathcal{H}_{\rm BdG} \begin{pmatrix} c_{\mathbf{k}} \\ c^\dagger_{-\mathbf{k}} \end{pmatrix} \qquad \mathcal{H}_{\rm BdG} = \mathbf{m}(\mathbf{k})\cdot\vec{\tau}$$

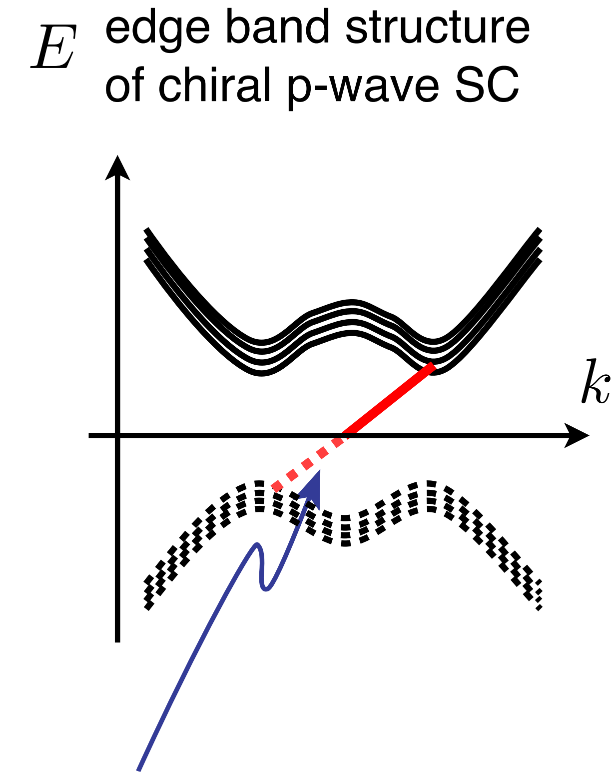
$$m_x(\mathbf{k}) = \Delta_0 \sin k_x \qquad m_y(\mathbf{k}) = \Delta_0 \sin k_y \qquad m_z(\mathbf{k}) = 2t\left[\cos k_x + \cos k_y\right] - \mu$$

Particle-hole symmetry: $\tau_x \mathcal{H}^*_{\rm BdG}(\mathbf{k})\tau_x = -\mathcal{H}_{\rm BdG}(-\mathbf{k})$

$(\tau_x)^2 = 1 \quad \Rightarrow \quad \text{class D}$

$$E = \pm\,|\boldsymbol{m}(\boldsymbol{k})|$$

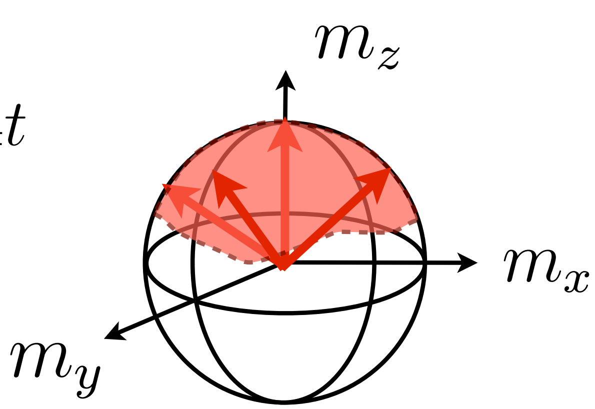
Spectrum flattening: $\hat{\boldsymbol{m}}(\boldsymbol{k}) = \dfrac{\boldsymbol{m}(\boldsymbol{k})}{|\boldsymbol{m}(\boldsymbol{k})|}$



trivial phase

$|\mu| > 4t$

$n = 0$

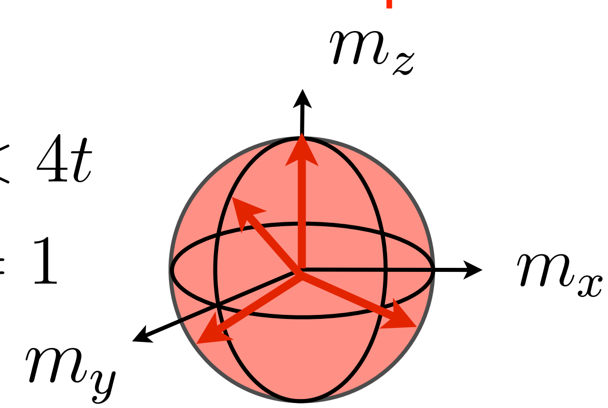


no edge state

non-trivial phase

$|\mu| < 4t$

$n = 1$



chiral Majorana edge state

classified by
Chern number:
(winding number)

$$n = \frac{1}{8\pi}\int_{\rm BZ} d^2\mathbf{k}\,\epsilon^{\mu\nu}\,\hat{\mathbf{m}}\cdot\left[\partial_{k_\mu}\hat{\mathbf{m}}\times\partial_{k_\nu}\hat{\mathbf{m}}\right]$$

Sr_2RuO_4 (n=2)

Mapping $\quad \hat{\boldsymbol{m}}(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $\hat{\boldsymbol{m}}(\boldsymbol{k}) \in S^2 \qquad$ "$\pi_2(S^2) = \mathbb{Z}$"

Majorana fermions in chiral p-wave superconductor

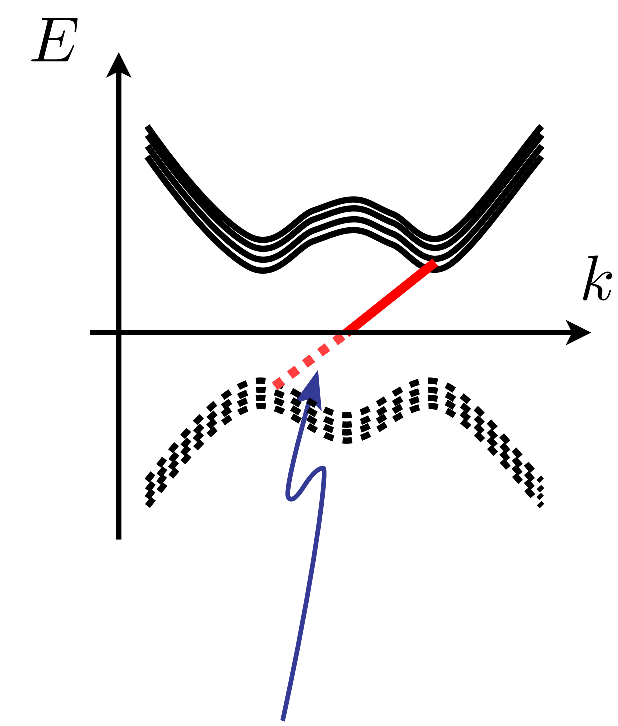
- ▶ Bulk-boundary correspondence: $n = \#$ Majorana edge modes

Majorana edge states are perfect heat conductor

⟹ Quantized thermal Hall conductance

$$\frac{\kappa_{xy}}{T} = \frac{\pi k_B^2}{48h} \int_{\mathrm{BZ}} d^2\mathbf{k}\, \epsilon^{\mu\nu} \hat{\mathbf{m}} \cdot \left[\partial_{k_\mu} \hat{\mathbf{m}} \times \partial_{k_\nu} \hat{\mathbf{m}}\right]$$

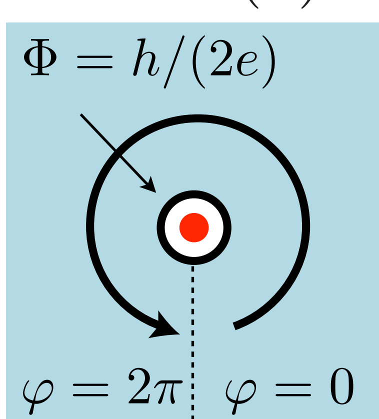
edge band structure of chiral p-wave SC



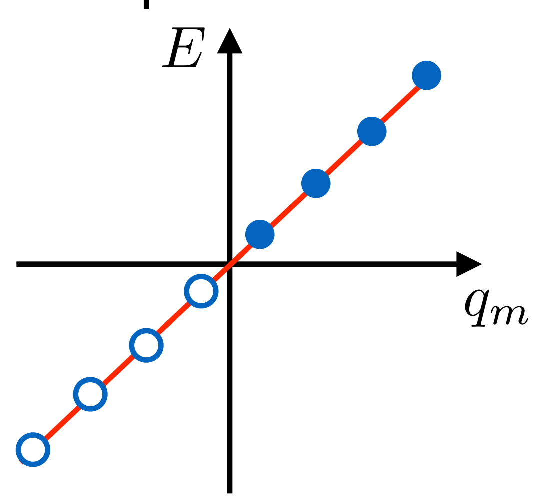
Majorana state

- ▶ Majorana zero mode at a vortex:
  - vortex: small hole with edge states
  - Majorana zero mode for $\Phi = p\dfrac{h}{2e}$ with p odd
    (periodic vs. anti-periodic BC)

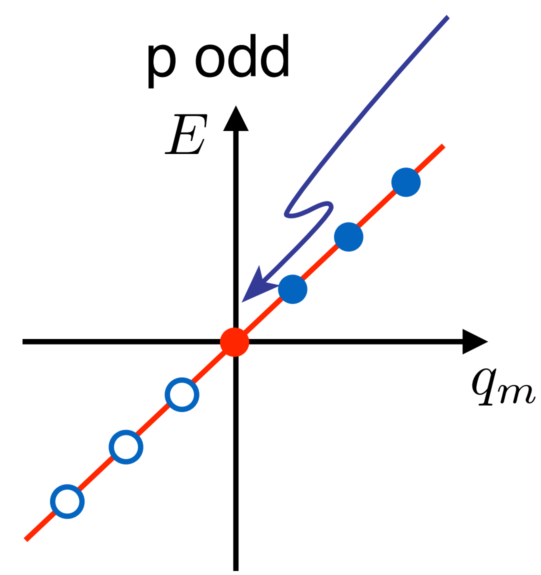
$$\Delta = \Delta(r) e^{i\varphi}$$



p even



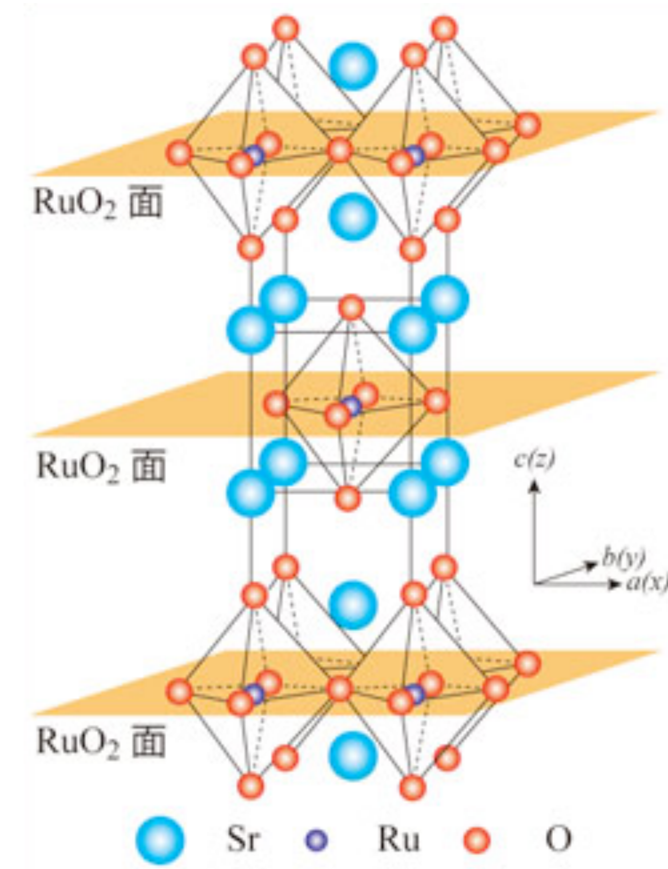
p odd



[Caroli, de Gennes, Matricon '64]

Experimental detection of spinful chiral p-wave SC

The transition-metal-oxide Sr_2RuO_4 is likely (?) a *spinful* chiral p-wave superconductor with Chern number n=2 (per layer)



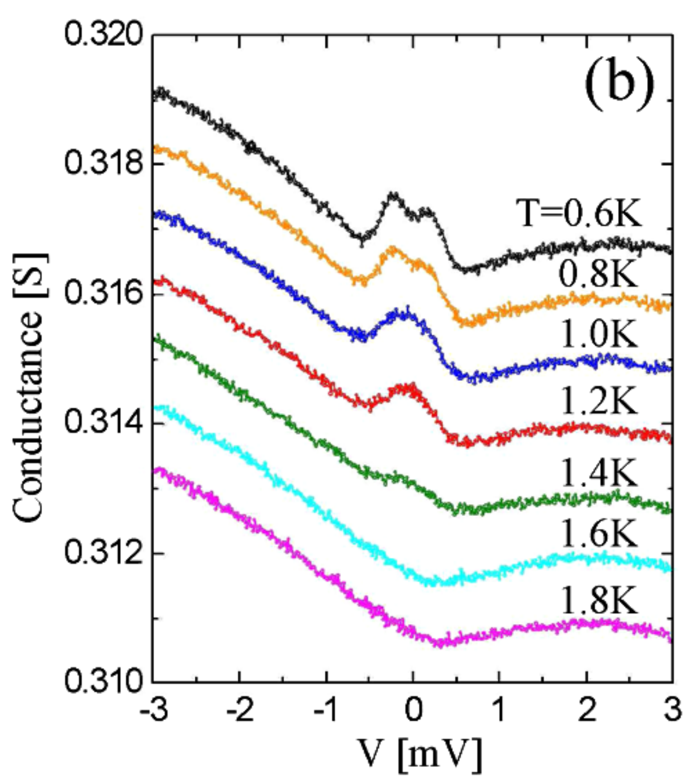
— Ru t_{2g}-orbitals ($4d^4$-electrons) hybridized with O p-oribitals form quasi-two-dimensional Fermi surfaces

— transition temperature T_C = 1.5K

— strong anisotropies in spin dependent responses (NMR and Knight shift)

— signatures of edge states in tunneling conductance

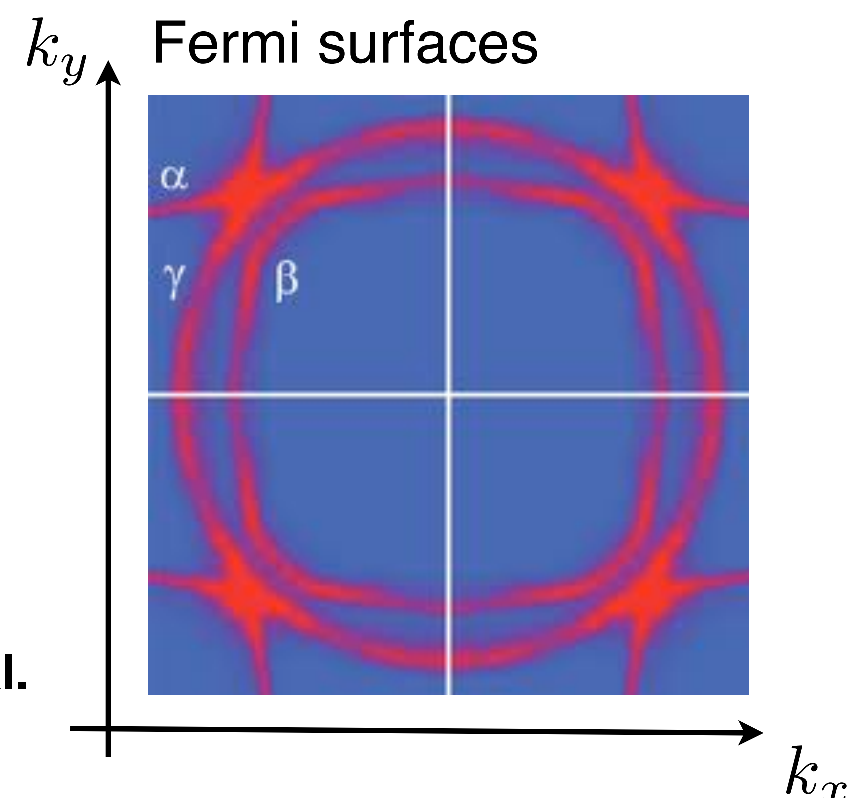
tunneling conductance



[Maeno et al. JPSJ 81, 011009]

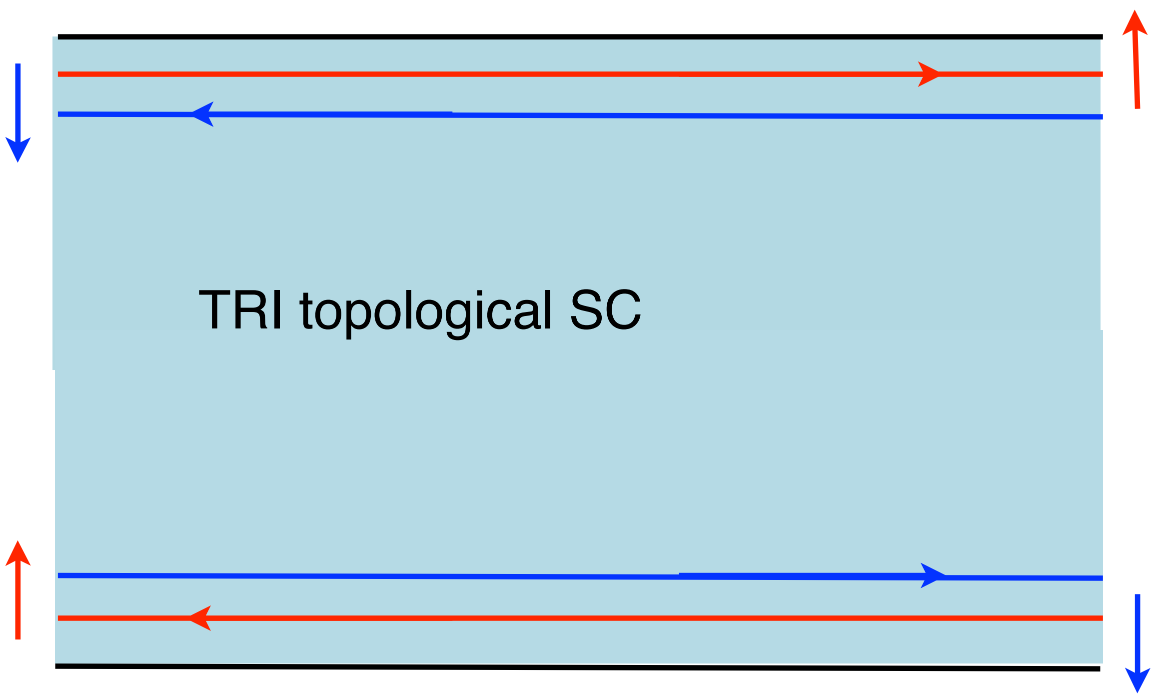
[Kashiwaya et al. PRL 2011]

Fermi surfaces



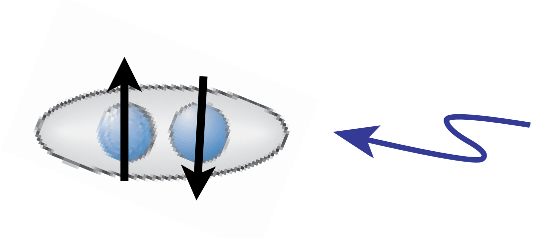
[Damascelli et al. PRL 2000]

Helical superconductors
(w/ time-reversal symmetry)



Time-reversal-invariant topological superconductor

Superconducting pairing with spin:



Cooper pair

$$H_{\mathrm{MF}} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\sigma\sigma'} \left[\Delta_{\sigma\sigma'}(\mathbf{k}) c^{\dagger}_{\mathbf{k},\sigma} c^{\dagger}_{-\mathbf{k},\sigma'} + \Delta^{*}_{\sigma\sigma'}(\mathbf{k}) c_{-\mathbf{k},\sigma'} c_{\mathbf{k},\sigma} \right]$$

2 x 2 Gap matrix: $\Delta(\mathbf{k}) = [\Delta_s(\mathbf{k})\sigma_0 + \mathbf{d}(\mathbf{k}) \cdot \vec{\sigma}] i\sigma_y$

Time-reversal symmetry: $\sigma_y \Delta^{\dagger}(\mathbf{k}) \sigma_y = \Delta^{\mathrm{T}}(-\mathbf{k})$

Different spin-pairing symmetries: (anti-symmetry of wavefunction)

spin-singlet: $\Delta_s(\mathbf{k}) : \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ even parity: $\Delta_s(\mathbf{k}) = \Delta_s(-\mathbf{k})$

spin-triplet:
$$\begin{cases} d_x(\mathbf{k}) - i d_y(\mathbf{k}) : \ |\uparrow\uparrow\rangle \\ d_x(\mathbf{k}) + i d_y(\mathbf{k}) : \ |\downarrow\downarrow\rangle \\ d_z(\mathbf{k}) : \ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \end{cases}$$

odd parity: $\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$

2D time-reversal-invariant topological superconductor

(also known as “helical superconductor”)

Square lattice BdG Hamiltonian in the presence of time-reversal symmetry:

Simplest model:
(spinless chiral p-wave SC)^2

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \mathcal{H}_{p+ip}(\mathbf{k}) & 0 \\ 0 & \mathcal{H}_{p-ip}(\mathbf{k}) \end{pmatrix}$$

$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y) - \mu \qquad d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = 0$$

TRS: $\boxed{T\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})T^{-1} = +\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})}$ $\quad T = i\sigma_y \otimes \tau_0 \mathcal{K} \quad T^2 = -1$

PHS: $\boxed{C\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})C^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})}$ $\quad C = \sigma_0 \otimes \tau_x \mathcal{K} \quad C^2 = +1$

} class DIII

▶ Combination of time-reversal and particle-hole symmetry:

(chiral symmetry) $\quad U_S = (i\sigma_y \otimes \tau_0)(\sigma_0 \otimes \tau_x) \qquad U_S\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) + \mathcal{H}_{\mathrm{BdG}}(\mathbf{k})U_S = 0$

▶ $\mathcal{H}_{\mathrm{BdG}}$ can be brought into block-off diagonal form: (transform to basis in which S is diagonal)

$$\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \qquad D(\mathbf{k}) = (i\sigma_y)\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}]\}$$

▶ TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

2D time-reversal-invariant topological superconductor

(also known as “helical superconductor”)

Square lattice BdG Hamiltonian in the presence of time-reversal symmetry:

Simplest model:
(spinless chiral p-wave SC)²

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y) - \mu \qquad d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = 0$$

$$\left.\begin{array}{llll} \text{TRS:} & \boxed{T\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})T^{-1} = +\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})} & T = i\sigma_y \otimes \tau_0 \mathcal{K} & T^2 = -1 \\ \text{PHS:} & \boxed{C\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})C^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})} & C = \sigma_0 \otimes \tau_x \mathcal{K} & C^2 = +1 \end{array}\right\} \text{class DIII}$$

- ▶ Combination of time-reversal and particle-hole symmetry:

(chiral symmetry) $U_S = (i\sigma_y \otimes \tau_0)(\sigma_0 \otimes \tau_x)$ $\quad U_S\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) + \mathcal{H}_{\mathrm{BdG}}(\mathbf{k})U_S = 0$

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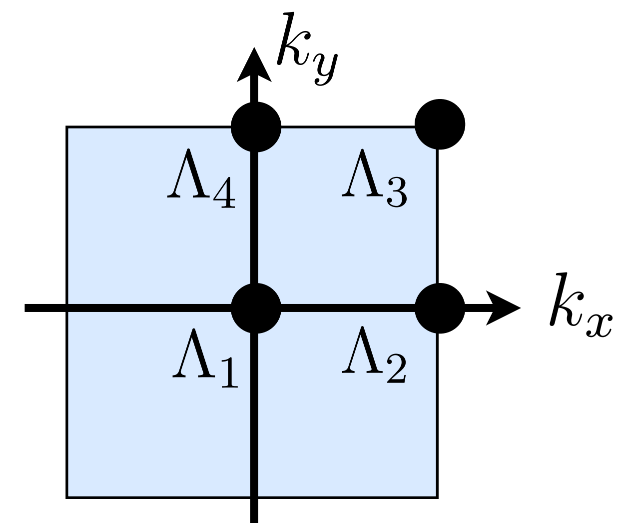
- ▶ TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

2D time-reversal-invariant topological superconductor

$$\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \quad \text{where:} \quad D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\right\}$$



- Spectrum flattening:

Projector onto filled Bloch bands

$$Q = \mathbb{1}_{4N} - 2P \qquad Q(\mathbf{k}) = \begin{pmatrix} 0 & q(\mathbf{k}) \\ q^\dagger(\mathbf{k}) & 0 \end{pmatrix}$$

- TRS acts on $q(\mathbf{k})$ as follows: $\quad q(\mathbf{k}) = -q^T(-\mathbf{k})$

- The eigenfunctions of $Q(\mathbf{k})$ are:

$$|u_a^\pm(\boldsymbol{k})\rangle_{\mathrm{N}} = \frac{1}{\sqrt{2}}\begin{pmatrix} n_a \\ \pm q^\dagger(\boldsymbol{k}) n_a \end{pmatrix} \quad \text{where:} \quad (n_a)_b = \delta_{ab}$$

are globally defined.

Z_2 topological invariant:

sewing matrix

$$(-1)^\nu = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[\omega(\Lambda_a)\right]}{\sqrt{\det\left[\omega(\Lambda_a)\right]}} = \pm 1 \qquad \omega(\mathbf{k}) = {}_{\mathrm{N}}\langle u_a^-(-\mathbf{k})|\Theta u_b^-(\mathbf{k})\rangle_{\mathrm{N}}$$

$$\Rightarrow \boxed{(-1)^\nu = \prod_{a=1}^{4} \frac{\mathrm{Pf}\left[q^T(\Lambda_a)\right]}{\sqrt{\det\left[q(\Lambda_a)\right]}} = \pm 1}$$

$$q(\mathbf{k}) = -q^T(-\mathbf{k})$$

$$q^\dagger(\mathbf{k}) = q^{-1}(\mathbf{k})$$

(same symmetries as sewing matrix)

2D time-reversal-invariant topological superconductor

Effective low-energy continuum theory:
(expand around $\mathbf{k} = 0$)

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

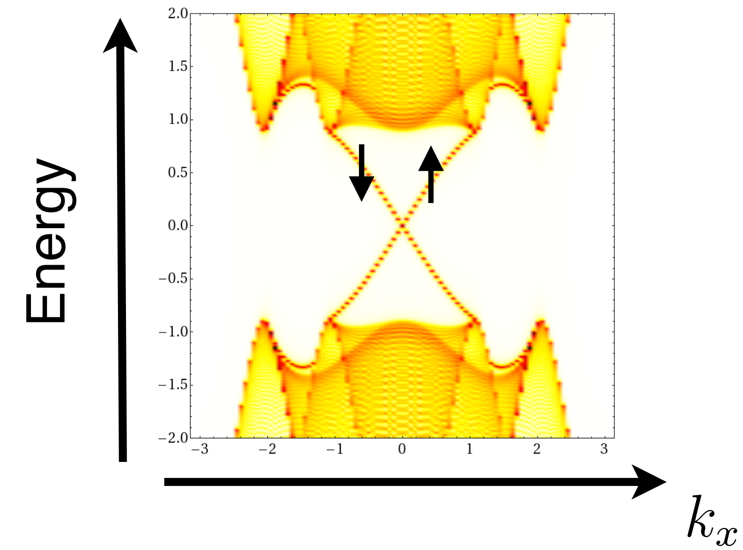
$$\varepsilon(\mathbf{k}) = -tk^2 + 4t - \mu \qquad d_x(\mathbf{k}) = k_x \qquad d_y(\mathbf{k}) = k_y \qquad d_z(\mathbf{k}) = 0$$

Energy spectrum: $E_\pm = \pm\lambda(\mathbf{k}) = \pm\sqrt{\varepsilon^2(\mathbf{k}) + \Delta_t(k_x^2 + k_y^2)}$ TRIM: $k = 0, \quad k = +\infty$

Z_2 topological invariant : $(-1)^\nu = -\mathrm{sgn}(4t - \mu)\mathrm{sgn}(t)$

$\mu > 4t$: trivial super-conductor

$\mu < 4t$: TRI topological superconductor



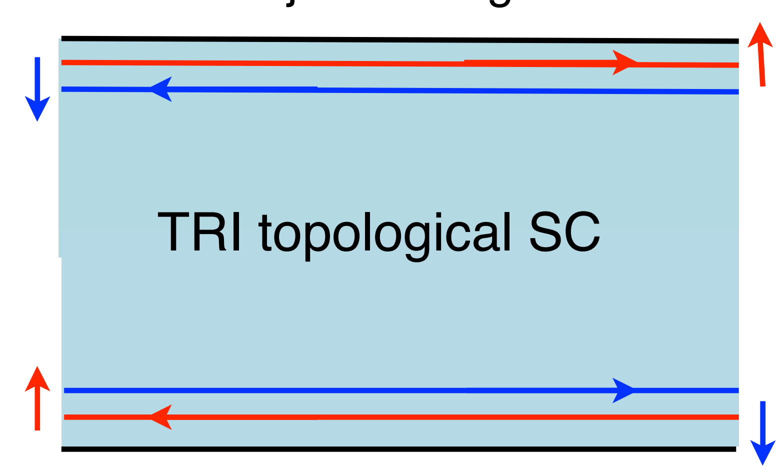
Bulk-boundary correspondence:

By analogy to chiral p-wave SC: (for $|\mu| < 4t$)
two counter-propagating Majorana edge modes

— protected by TRS and PHS

— possible condensed matter realization:
thin film of $CePt_3Si$?

helical Majorana edge states:

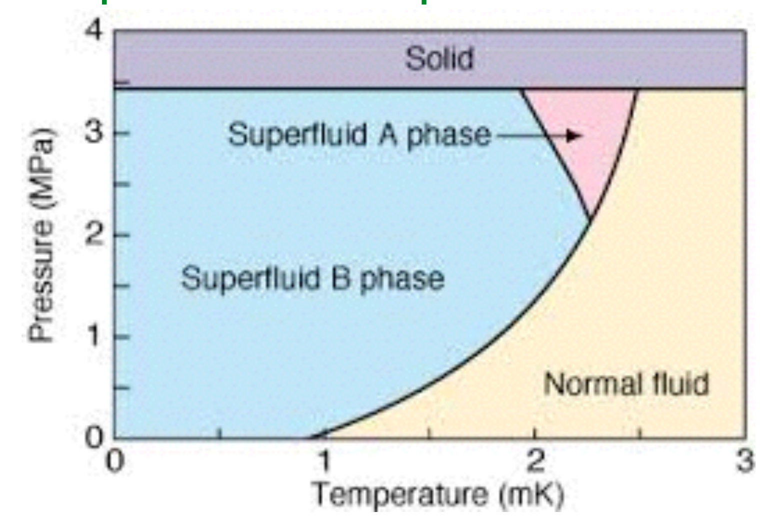


3D time-reversal-invariant topological superconductor

Cubic lattice BdG Hamiltonian in the presence of time-reversal symmetry:

$$\mathcal{H}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \Delta_t[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}](i\sigma_y) \\ \Delta_t(-i\sigma_y)[\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}] & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix}$$

equivalent to B phase of ^{3}He



$$\varepsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y + \cos k_z) - \mu$$

$$d_x(\mathbf{k}) = \sin k_x \quad d_y(\mathbf{k}) = \sin k_y \quad d_z(\mathbf{k}) = \sin k_z$$

TRS $\boxed{T\mathcal{H}_{\rm BdG}(\mathbf{k})T^{-1} = +\mathcal{H}_{\rm BdG}(-\mathbf{k})}$ $\quad T = i\sigma_y \otimes \tau_0 \mathcal{K} \quad T^2 = -1$

PHS $\boxed{C\mathcal{H}_{\rm BdG}(\mathbf{k})C^{-1} = -\mathcal{H}_{\rm BdG}(-\mathbf{k})}$ $\quad C = \sigma_0 \otimes \tau_x \mathcal{K} \quad C^2 = +1$

class DIII

Chiral symmetry (TRS x PHS): $U_S\mathcal{H}_{\rm BdG}(\mathbf{k}) + \mathcal{H}_{\rm BdG}(\mathbf{k})U_S = 0$

▶ $\mathcal{H}_{\rm BdG}$ can be brought into block-off diagonal form: (transform to basis in which S is diagonal)

$$\widetilde{\mathcal{H}}_{\rm BdG}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^\dagger(\mathbf{k}) & 0 \end{pmatrix} \qquad D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\right\}$$

▶ TRS acts on $D(\mathbf{k})$ as follows: $D^T(-\mathbf{k}) = -D(\mathbf{k})$

3D time-reversal-invariant topological superconductor

Lattice BdG Hamiltonian: $\widetilde{\mathcal{H}}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} 0 & D(\mathbf{k}) \\ D^{\dagger}(\mathbf{k}) & 0 \end{pmatrix}$

▶ Off-diagonal block: $\boxed{D(\mathbf{k}) = (i\sigma_y)\left\{\varepsilon_{\mathbf{k}}\sigma_0 + i\Delta_t[\mathbf{d}_{\mathbf{k}}\cdot\vec{\sigma}]\right\}}$

Mapping $D(\mathbf{k})$: Brillouin zone $\longmapsto$ $D(\mathbf{k})$ TRS: $D(\mathbf{k}) = -D^T(\mathbf{k})$

▶ Spectrum flattening: $q(\mathbf{k}) = \sum_a \frac{1}{\lambda_a(\mathbf{k})} u_a(\mathbf{k}) u_a^{\dagger}(\mathbf{k}) D(\mathbf{k})$ $u_a(\boldsymbol{k})$: eigenvectors of $DD^{\dagger}$

Mapping $q(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $q(\boldsymbol{k}) \in U(2)$

TRS: $q(\mathbf{k}) = -q^T(-\mathbf{k})$

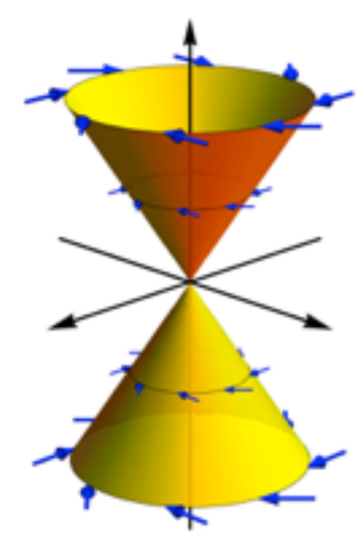
$\pi_2[U(2)] = 0$

$\pi_3[U(2)] = \mathbb{Z}$

$\Longrightarrow$ classified by winding number: $\boxed{W = \frac{1}{24\pi^2}\int_{\mathrm{BZ}} d^3k\, \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]}$

▶ Bulk-boundary correspondence:

$|W| = \#$ Kramers-degenerate Majorana states



Possible condensed matter realization:
$CePt_3Si$, Li_2Pt_3B, $CeRhSi_3$, $CeIrSi_3$, etc.