Topological insulators and superconductors

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1. Symmetries & ten-fold classification

   - Symmetry classes of ten-fold way
   - Dirac Hamiltonians and Dirac mass gaps
   - Periodic table of topological insulators and superconductors

2. Topological semi-metals and nodal superconductors

   - Momentum-dependent invariants
   - Examples: Weyl semi-metal, Weyl superconductors, etc.
   - Classification in terms of global symmetries

Symmetry classes: "Ten-fold way"

(originally introduced in the context of random Hamiltonians / matrices)

time-reversal invariance: $T = U_T \mathcal{K}$ (is antiunitary)

$$T^{-1}\mathcal{H}(-\mathbf{k})T = +\mathcal{H}(\mathbf{k})$$

$$T : \begin{cases} 0 & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and} \quad T^2 = +1 \\ -1 & \text{time reversal invariance and} \quad T^2 = -1 \end{cases}$$

complex conjugation

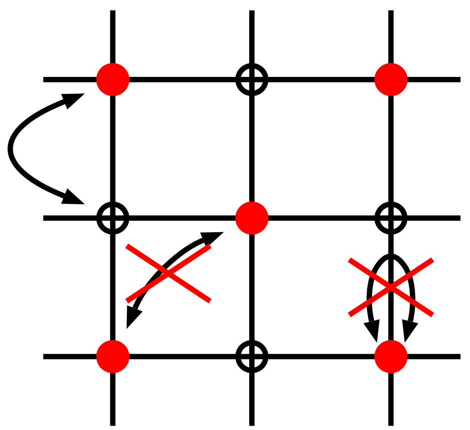
particle-hole symmetry (Ξ): $C = U_C \mathcal{K}$

$$C^{-1}\mathcal{H}(-\mathbf{k})C = -\mathcal{H}(\mathbf{k})$$

$$C : \begin{cases} 0 & \text{no particle-hole symmetry} \\ +1 & \text{particle-hole symmetry and} \quad C^2 = +1 \\ -1 & \text{particle-hole symmetry and} \quad C^2 = -1 \end{cases}$$

In addition we can also consider the "sublattice symmetry" $S \propto TC$

S : $S\mathcal{H}(\mathbf{k}) + \mathcal{H}(\mathbf{k})S = 0$



Note: SLS is often also called "chiral symmetry"

Ten-fold classification of topological insulators and superconductors

Ten-fold classification:

— classifies fully gapped topological materials in terms of *non-spatial symmetries* (i.e., symmetries that act *locally* in space)

— non-spatial symmetries:

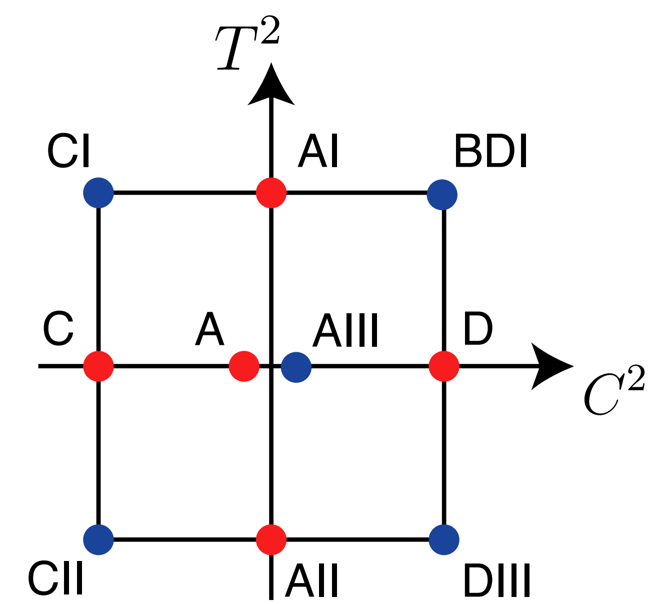
- time-reversal: $T\mathcal{H}(\mathbf{k})T^{-1} = +\mathcal{H}(-\mathbf{k}); \qquad T^2 = \pm 1$
- particle-hole: $C\mathcal{H}(\mathbf{k})C^{-1} = -\mathcal{H}(-\mathbf{k}); \qquad C^2 = \pm 1$
- sublattice: $S\mathcal{H}(\mathbf{k})S^{-1} = -\mathcal{H}(\mathbf{k}); \qquad S \propto TP$

ten symmetry classes

Altland-Zirnbauer Random Matrix Classes

Symmetry Class	T	C	S
A	0	0	0
AIII	0	0	1
AI	1	0	0
BDI	1	1	1
D	0	1	0
DIII	-1	1	1
AII	-1	0	0
CII	-1	-1	1
C	0	-1	0
CI	1	-1	1

"Bott clock"



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? For which symmetry class and dimension is there a topological insulator/superconductor?

Symmetries and Dirac Hamiltonians

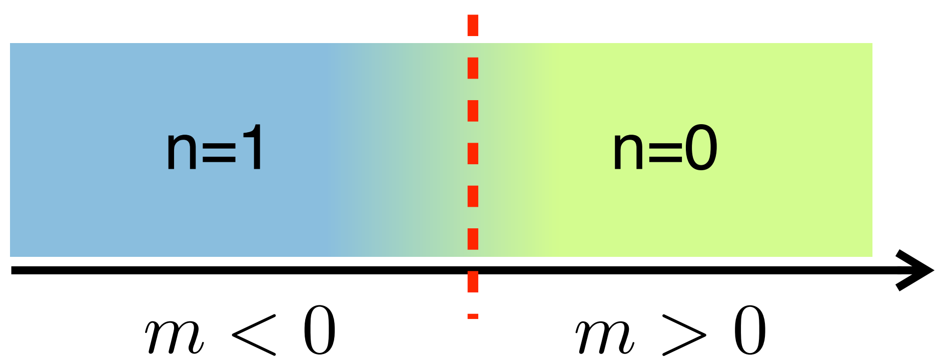
Dirac Hamiltonian in spatial dimension d : $\mathcal{H}(k) = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$ $\quad E_{\pm} = \pm\sqrt{m^2 + \sum_{i=1}^{d} k_i^d}$

- Gamma matrices γ_i obey: $\{\gamma_i, \gamma_j\} = 2\delta_{ij} \qquad i = 0, 1, \ldots, d$

- TRS, PHS and chiral symmetry lead to the conditions:

$$[\gamma_0, T] = 0 \qquad \{\gamma_{i\neq 0}, T\} = 0$$
$$\{\gamma_0, C\} = 0 \qquad [\gamma_{i\neq 0}, C] = 0$$
$$\{\gamma_i, S\} = 0$$

- Topological phase transition as a function of mass term $m\gamma_0$



? are there extra symmetry preserving mass terms $M\gamma_{d+1}$ that connect the two phases without gap closing?

$$\{\gamma_{d+1}, \gamma_i\} = 0 \quad i = 0, 1, \cdots 2 \qquad E_{\pm} = \pm\sqrt{m^2 + M^2 + \sum_{i=1}^{d} k_i^d}$$

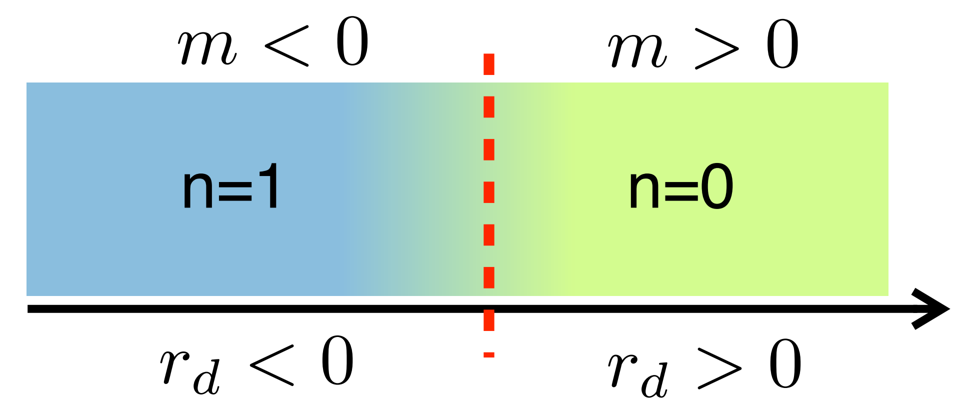
NO: topologically non-trivial **YES:** topologically trivial

Symmetries and Dirac Hamiltonians

Dirac Hamiltonian in spatial dimension d : $\mathcal{H}(k) = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$ $E_\pm = \pm\sqrt{m^2 + \sum_{i=1}^{d} k_i^d}$

- Gapless surface states (interface states): $k_d \to i\partial/\partial r_d$

$$\mathcal{H} = \gamma_0\left(\widetilde{m}\mathbb{I} - i\gamma_0\gamma_d \frac{\partial}{\partial r_d}\right) + \sum_{i=1}^{d-1} k_i \gamma_i$$



surface state ϕ: $i\gamma_0\gamma_d \Phi = \pm\Phi$

surface Hamiltonian: $\mathcal{H}_{\text{surf}} = \sum_{i=1}^{d-1} k_i \, \mathbf{P}\gamma_i\mathbf{P}$

$$\mathbf{P} = \left(\mathbb{I} - i\gamma_0\gamma_d\right)/2$$

gapless surface spectrum: $E^{\pm}_{\text{surf}} = \pm\sqrt{\sum_{i=1}^{d-1} k_i^2}$

- Presence of extra symmetry preserving mass term implies gapped surface states

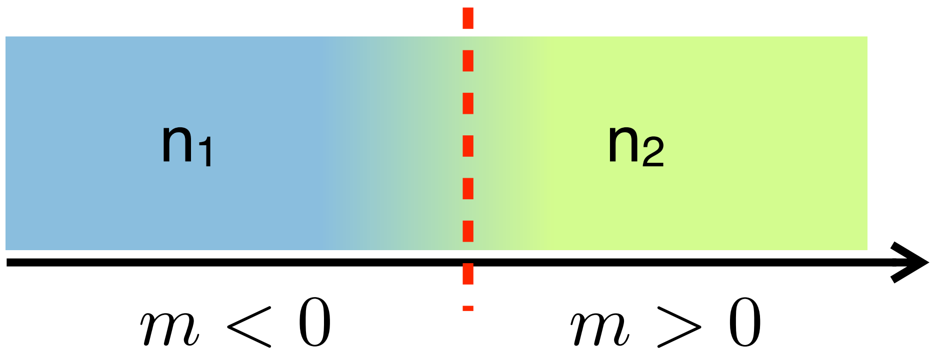
— extra mass term projected onto surface is non-vanishing

$M\mathbf{P}\gamma_{d+1}\mathbf{P}$ anti-commutes with $\mathbf{P}\gamma_i\mathbf{P}$ $\quad i = 1, \ldots, d-1$

$\Longrightarrow$ gapped surface spectrum

Dirac Hamiltonian in symmetry class AIII

- Topological phase transition as a function of mass term $m\gamma_0$



$$S = \sigma_1 \qquad S\mathcal{H}(\mathbf{k}) + \mathcal{H}(\mathbf{k})S = 0$$

- *One-dimensional* Dirac Hamiltonian with rank 2:

$$\mathcal{H}(k) = k\sigma_3 + m\sigma_2$$

— no extra symmetry preserving mass term exists

$\Longrightarrow$ class AIII in 1D is topologically non-trivial

— space of normalized mass matrices $V^{\mathrm{AIII}}_{d=1,r=2} = \{\pm\sigma_2\}$

One-dimensional Dirac Hamiltonian in symmetry class AII

$$T^{-1}\mathcal{H}(-\mathbf{k})T = +\mathcal{H}(\mathbf{k}) \qquad T^2 = -1$$

✴Dirac matrices with rank 2:

$$\mathcal{H}(k) = k\sigma_3 \qquad\qquad T = i\sigma_2\mathcal{K}$$

— no symmetry-allowed mass term exists $\Rightarrow$ impossible to localize
$(\sigma_1 \text{ and } \sigma_2 \text{ violate TRS})$

— describes edge state of 2D topological insulator in class AII

✴Dirac matrices with rank 4:

$$\mathcal{H}(k) = k\sigma_3 \otimes \tau_1 + m\sigma_0 \otimes \tau_3 \qquad\qquad T = i\sigma_2 \otimes \tau_0\mathcal{K}$$

— extra symmetry preserving mass term: $\quad M\sigma_3 \otimes \tau_2$

$\Longrightarrow$ class AII in 1D is topologically trivial

— space of normalized mass matrices

$$V^{\mathrm{AII}}_{d=1,r=4} = \{\mathbf{M}\cdot\mathbf{X}|\ \mathbf{M}^2 = 1\} = S^1 \qquad\qquad R_3 : U(2N)/Sp(N)$$

$$\mathbf{M} = (m, M), \qquad \mathbf{X} = (\sigma_0 \otimes \tau_3, \sigma_3 \otimes \tau_2)$$

• connectedness of space of normalized Dirac masses: $\quad \pi_0(R_3) = 0$

Two-dimensional Dirac Hamiltonian in symmetry class AII

$$T^{-1}\mathcal{H}(-\mathbf{k})T = +\mathcal{H}(\mathbf{k}) \qquad T^2 = -1$$

- Dirac matrices with rank 4: $T = i\sigma_2 \otimes \tau_0 \mathcal{K}$

$$\mathcal{H}(\mathbf{k}) = k_1\sigma_3 \otimes \tau_1 + k_2\sigma_0 \otimes \tau_2 + m\sigma_0 \otimes \tau_3$$

— no symmetry-allowed mass term exists $\Rightarrow$ topologically non-trivial

$(\sigma_1 \otimes \tau_1,\ \sigma_2 \otimes \tau_1 \ \text{violate TRS})$

- "Doubled" Dirac Hamiltonian:

$$\mathcal{H}_2(\mathbf{k}) = \begin{pmatrix} \mathcal{H}(\mathbf{k}) & 0 \\ 0 & \hat{\mathcal{H}}_{\mu\nu\lambda}(\mathbf{k}) \end{pmatrix} \qquad \mu, \nu, \lambda \in \{+1, -1\}$$

$$\hat{\mathcal{H}}_{\mu\nu\lambda}(\mathbf{k}) = \mu k_1\sigma_3 \otimes \tau_1 + \nu k_2\sigma_0 \otimes \tau_2 + \lambda m\sigma_0 \otimes \tau_3$$

— extra symmetry preserving mass terms:

e.g. for $\mu = +,\ \nu = +,\ \lambda = +$: $\sigma_2 \otimes \tau_1 \otimes s_1,\ \sigma_1 \otimes \tau_2 \otimes s_2$

$\Longrightarrow$ gapped surface spectrum

$\Longrightarrow$ class AII in 2D has Z_2 classification

— space of normalized mass matrices: $R_2 = O(2N)/U(N) \qquad \pi_0(R_2) = \mathbb{Z}_2$

Dirac Hamiltonian in symmetry class A

- *One-dimensional* Dirac Hamiltonian with rank 2:

$$\mathcal{H}(k) = k\sigma_1 + m\sigma_2 + \mu\sigma_0$$

— extra symmetry preserving mass term: $M\sigma_3$

$\Longrightarrow$ class A in 1D is topologically trivial

— space of normalized mass matrices

$$V^{\mathrm{A}}_{d=1,r=2} = \{\tau_2 \cos\theta + \tau_3 \sin\theta | 0 \leq \theta < 2\pi\} = S^1 \qquad C_1 : U(N)$$

— connectedness of space of normalized Dirac masses: $\pi_0(C_1) = 0$

- *Two-dimensional* Dirac Hamiltonian with rank 2:

$$\mathcal{H}(\mathbf{k}) = k_x\sigma_x + k_y\sigma_y + m\sigma_z + \mu\sigma_0$$

— no extra mass term exists $\Longrightarrow$ class A in 2D is topologically non-trivial

— describes two-dimensional Chern insulator

- *Two-dimensional* “doubled” Dirac Hamiltonian:

$$\mathcal{H}_2(\mathbf{k}) = \mathcal{H}(\mathbf{k}) \otimes \tau_0$$

— no extra gap opening mass term exists $\Longrightarrow$ topologically non-trivial

$\Rightarrow$ indicates $\mathbb{Z}$ classification

Homotopy classification of Dirac mass gaps

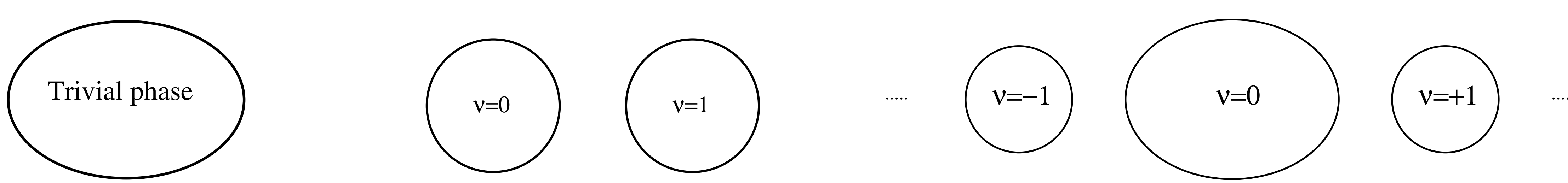
$\ast$ The space of mass matrices $V^{s}_{d,r=N}$ belongs to different classifying spaces C_{s-d} (for "complex class") or R_{s-d} (for "real class")

— the relation between AZ symmetry class and classifying space is as follows:

	classifying space	$\pi_0(*)$	1D AZ class	2D AZ class
$\mathcal{C}_0$	$\cup_{n=0}^{N}\{U(N)/[U(n)\times U(N-n)]\}$	$\mathbb{Z}$	AIII	A
$\mathcal{C}_1$	$U(N)$	0	A	AIII
$\mathcal{R}_0$	$\cup_{n=0}^{N}\{O(N)/[O(n)\times O(N-n)]\}$	$\mathbb{Z}$	BDI	D
$\mathcal{R}_1$	$O(N)$	$\mathbb{Z}_2$	D	DIII
$\mathcal{R}_2$	$O(2N)/U(N)$	$\mathbb{Z}_2$	DIII	AII
$\mathcal{R}_3$	$U(N)/Sp(N)$	0	AII	CII
$\mathcal{R}_4$	$\cup_{n=0}^{N}\{Sp(N)/[Sp(n)\times Sp(N-n)]\}$	$\mathbb{Z}$	CII	C
$\mathcal{R}_5$	$Sp(N)$	0	C	CI
$\mathcal{R}_6$	$Sp(2N)/U(N)$	0	CI	AI
$\mathcal{R}_7$	$U(N)/O(N)$	0	AI	BDI

$\ast$ The $0th$ homotopy group indexes the disconnected parts of the space of normalized mass matrices

$\pi_0(V)=0$ $\qquad$ $\pi_0(V)=\mathbb{Z}_2$ $\qquad$ $\pi_0(V)=\mathbb{Z}$



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ten symmetry classes

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CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	1	-1	1	0	0	$\mathbb{Z}$

Chern insulator

polyacetylene

2D topological insulator w/ SOC

3D topological insulator w/ SOC

$\mathbb{Z}$: integer classification
$\mathbb{Z}_2$: binary classification
0 : no topological state

Schnyder, Ryu, Furusaki, Ludwig, PRB (2008)

A. Kitaev, AIP (2009)

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	Symmetry			dim		
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CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
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CI	1	-1	1	0	0	$\mathbb{Z}$

chiral p-wave superconductor (Sr_2RuO_4)

TRI topological triplet SC (^3He B)

chiral d-wave superconductor

$\mathbb{Z}$: integer classification
$\mathbb{Z}_2$: binary classification
0 : no topological state

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Altland-Zirnbauer Random Matrix Classes

Symmetry				Spatial Dimension d								
Class	T	C	S	1	2	3	4	5	6	7	8	$\cdots$
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AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
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CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

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AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

- Topological invariants: Chern numbers and winding numbers

$$Ch_{n+1}[\mathcal{F}] = \frac{1}{(n+1)!} \int_{\mathrm{BZ}^{d=2n+2}} \mathrm{tr} \left(\frac{i\mathcal{F}}{2\pi} \right)^{n+1}$$

$$\nu_{2n+1}[q] = \frac{(-1)^n n!}{(2n+1)!} \left(\frac{i}{2\pi} \right)^{n+1} \int_{\mathrm{BZ}} \epsilon^{\alpha_1 \alpha_2 \cdots} \mathrm{tr} \left[q^{-1} \partial_{\alpha_1} q \cdot q^{-1} \partial_{\alpha_2} q \cdots \right] d^{2n+1} k$$

Extension I: Weak topological insulators and supercondutors

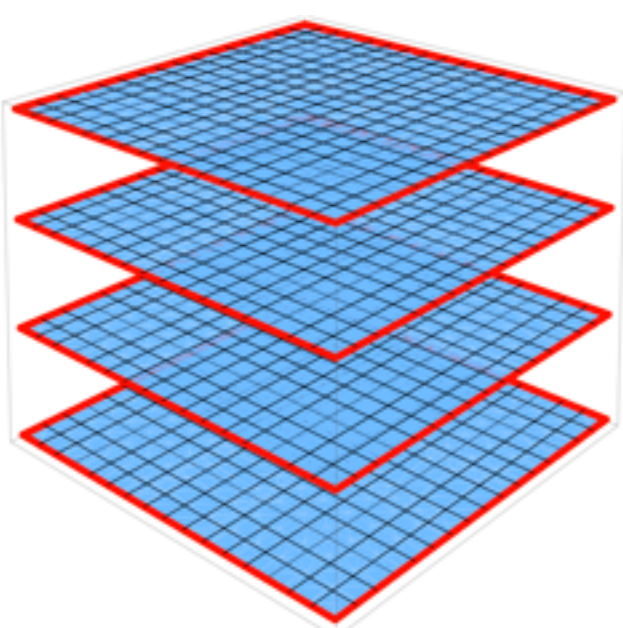
- **strong topological insulators (superconductors):**
  not destroyed by positional disorder

- **weak topological insulators (superconductors):**
  only possess topological features
  when translational symmetry is present

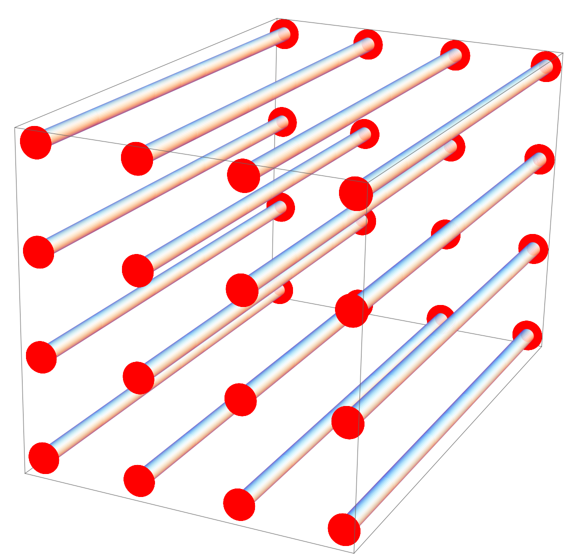
weak topological insulators (superconductors) are topologically equivalent to parallel stacks of lower-dimensional strong topological insulator (SCs).

	Symmetry			Dimension			
AZ	T	C	S	1	2	3	4
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	1	-1	1	0	0	$\mathbb{Z}$	0

co-dimension k=1



co-dimension k=2



d-dim.weak topological insulators (SCs) of co-dimension k can occur whenever there exists a strong topological state in same symmetry class but in (d-k) dimensions.

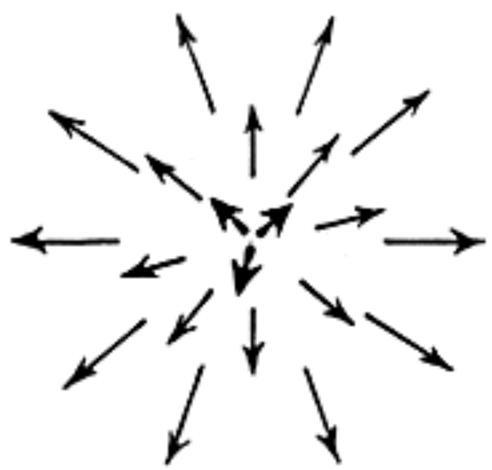
$\binom{d}{k}$ top. invariants $\qquad 0 < k \leq d$

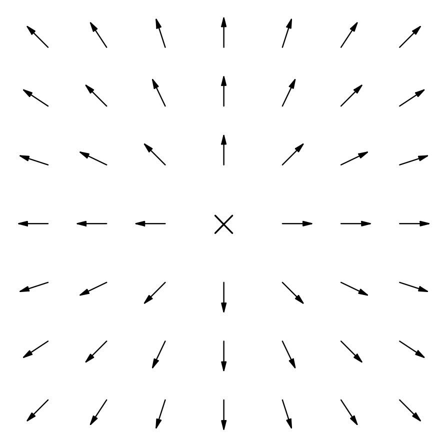
cf. Kitaev, AIP Conf Proc. 1134, 22 (2009)

Extension II: Zero mode localized on topological defect

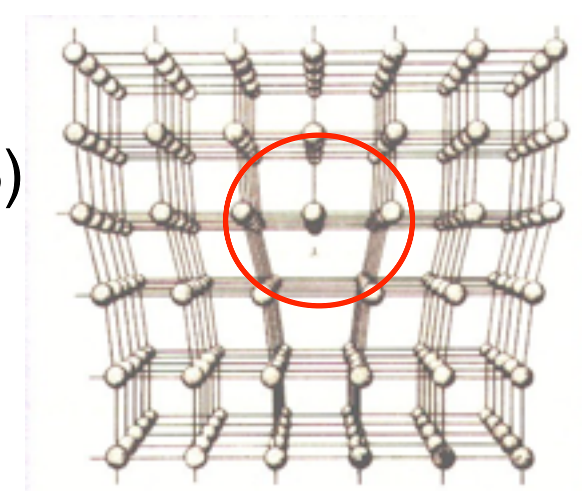
Protected zero modes can also occur at topological defects in D-dim systems

- Point defect (r=0): Hedgehog (D=3), vortex (D=2), domain wall (D=1)





- Line defect (r=1): dislocation line (D=3) domain wall (D=2)



- Two-dim defects (r=2): domain wall (D=3)

Freedman, et. al., PRB (2010)

Teo & Kane, PRB (2010)

Ryu, et al. NJP (2010)

	Symmetry			Dimension			
AZ	T	C	S	1	2	3	4
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	1	-1	1	0	0	$\mathbb{Z}$	0

Can an r-dimensional topological defect of a given symmetry class bind gapless states or not?

$\Longrightarrow$ look at column d=(r+1)

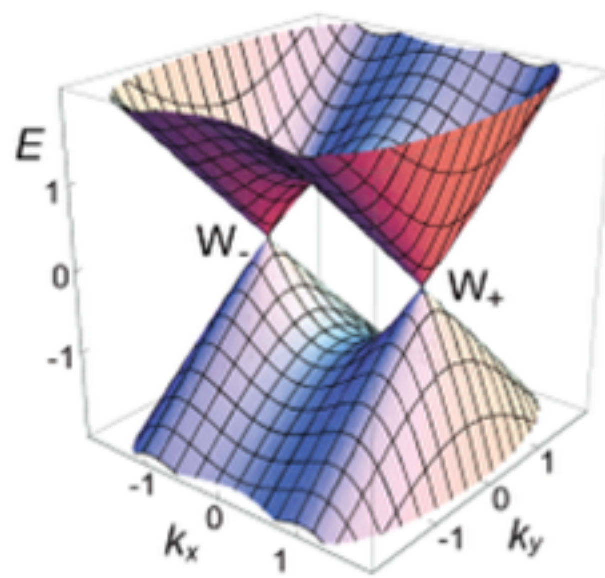
(answer does not depend on D!)

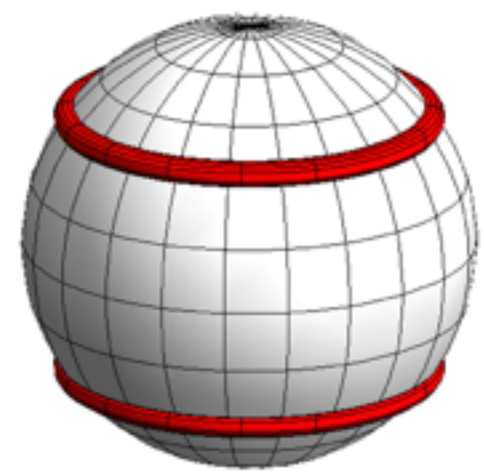
line defect in class A:

$$n = \frac{1}{8\pi^2}\int_{T^3\times S^1} \mathrm{Tr}[\mathcal{F}\wedge\mathcal{F}]$$

(second Chern no = no of zero modes)

Topological semi-metals and nodal superconductors





Topological nodal superconductors

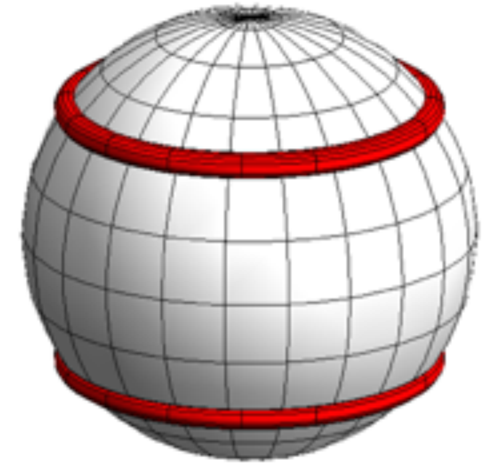
? How about topology of nodal superconductors and semi-metals?

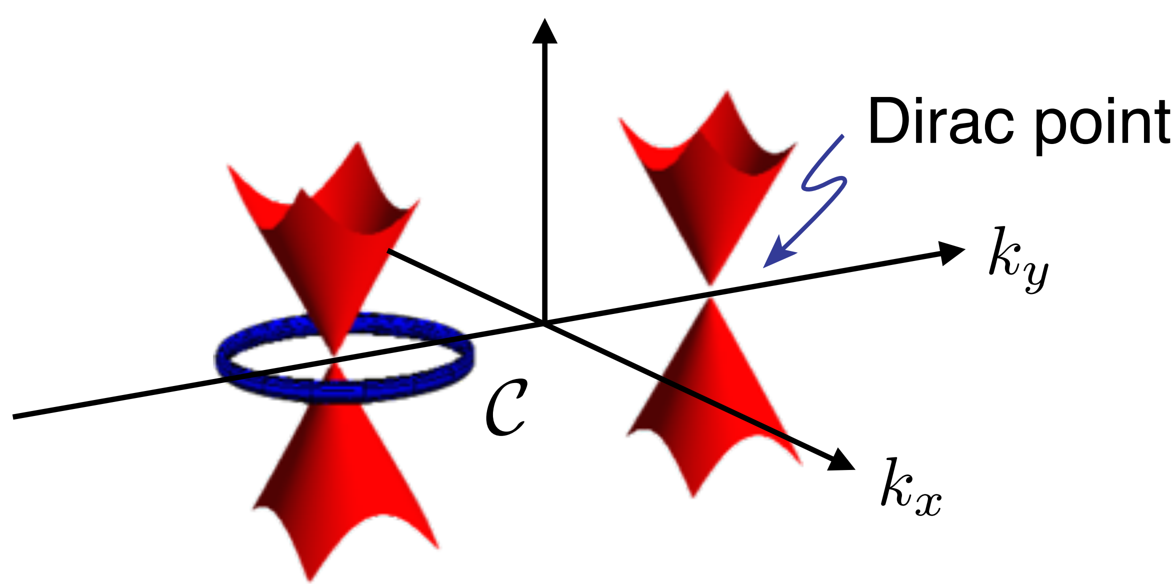
▶ Problem: Global topological number ill-defined (no gap!)

Solution: (assume translational symmetry)

⟹ Define **momentum-dependent topological number**

$$\boxed{W_{\mathcal{C}}^{\pm} = \frac{1}{2\pi} \oint_{\mathcal{C}} \omega_{\pm}(\mathbf{k}) dk_l}$$



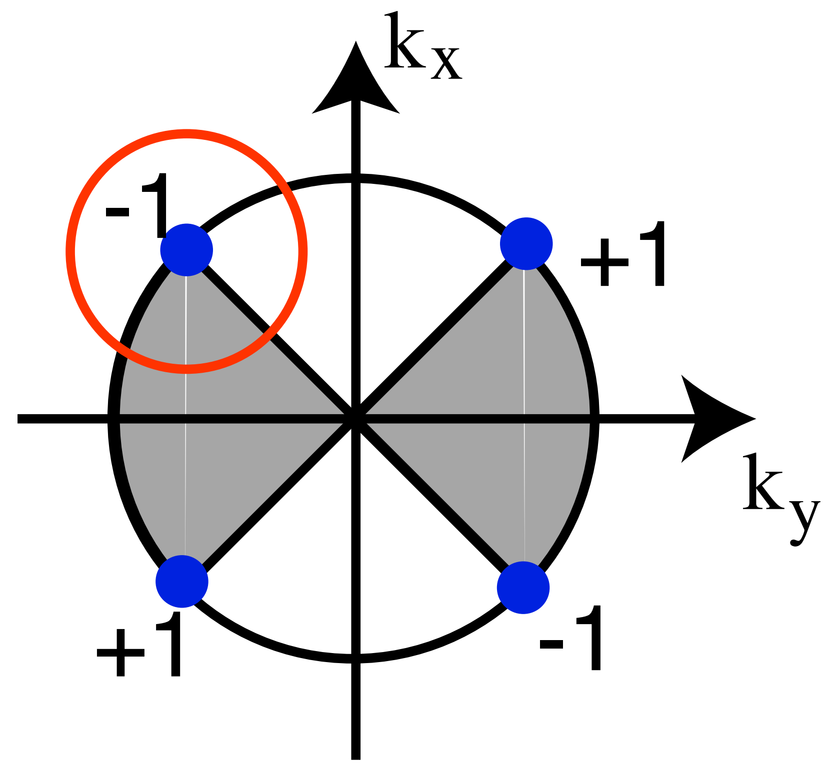


$\mathcal{C}$ *does* enclose Fermi point

$$W_{\mathcal{C}}^{\pm} = \pm 1 \quad \Rightarrow \text{topologically stable}$$

Topologically stable point nodes in d_{x2-y2} -wave SCs

- Consider $d_{x^2-y^2}$ -wave superconductor



$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} +\varepsilon_{\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta_{\boldsymbol{k}} & -\varepsilon_{\boldsymbol{k}} \end{pmatrix} \qquad \Delta_{\boldsymbol{k}} = \Delta_0(\cos k_x - \cos k_y)$$

Satisfies time-reversal symmetry *T* and particle-hole symmetry *C*

Combination of particle-hole symmetry and time-reversal symmetry gives

$$S\mathcal{H}(\boldsymbol{k})S^\dagger = -\mathcal{H}(\boldsymbol{k}) \qquad \text{with} \qquad S = TC = \sigma_2$$

In basis in which S is diagonal $\mathcal{H}(\boldsymbol{k})$ takes off-diagonal form: $\tilde{\mathcal{H}}(\boldsymbol{k}) = \begin{pmatrix} 0 & \varepsilon_{\boldsymbol{k}} - i\Delta_{\boldsymbol{k}} \\ \varepsilon_{\boldsymbol{k}} + i\Delta_{\boldsymbol{k}} & 0 \end{pmatrix}$

Spectrum flattening: $q(\boldsymbol{k}) = \dfrac{\varepsilon_{\boldsymbol{k}} + i\Delta_{\boldsymbol{k}}}{\sqrt{\varepsilon_{\boldsymbol{k}}^2 + \Delta_{\boldsymbol{k}}^2}}$ Consider: $q(\boldsymbol{k}) : S^1 \longrightarrow S^1$

$\pi_1(S^1) = \mathbb{Z}$ ⟹ nodal points are protected by one-dimensional *winding number:*

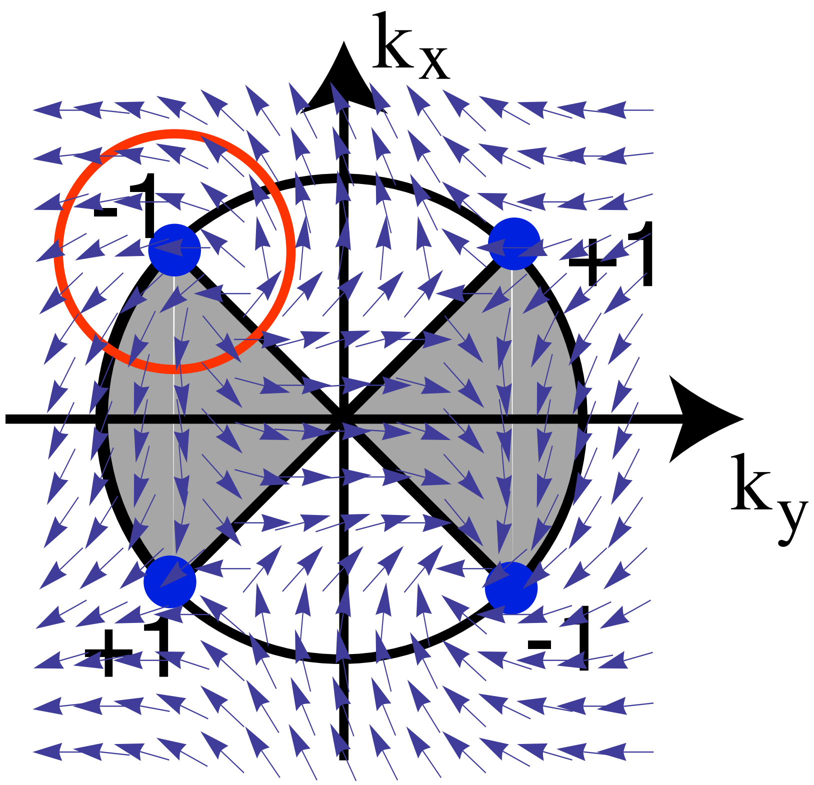
$$\boxed{W_{\mathcal{L}} = \frac{1}{2\pi i} \oint_{\mathcal{L}} dk_l \, \mathrm{Tr}\left[q^{-1}\partial_{k_l} q\right] = \pm 1}$$

Note: $W_{\mathcal{L}}$ is invariant under path deformation.

Topologically stable point nodes in d_{x2-y2} -wave SCs

- Consider $d_{x^2-y^2}$ -wave superconductor

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} +\varepsilon_{\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta_{\boldsymbol{k}} & -\varepsilon_{\boldsymbol{k}} \end{pmatrix} \qquad \Delta_{\boldsymbol{k}} = \Delta_0(\cos k_x - \cos k_y)$$



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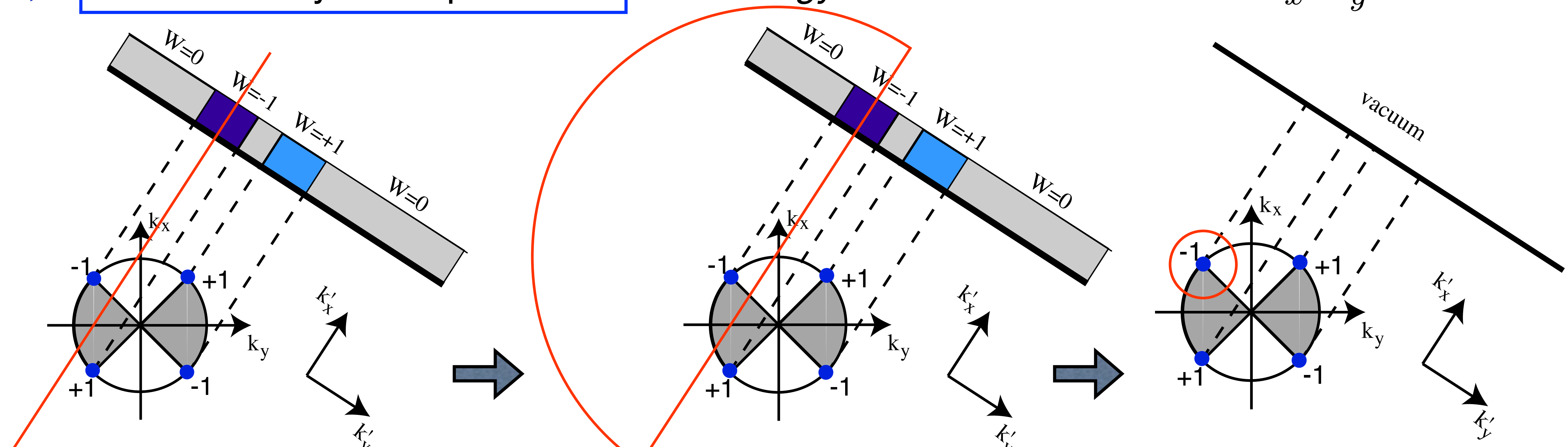
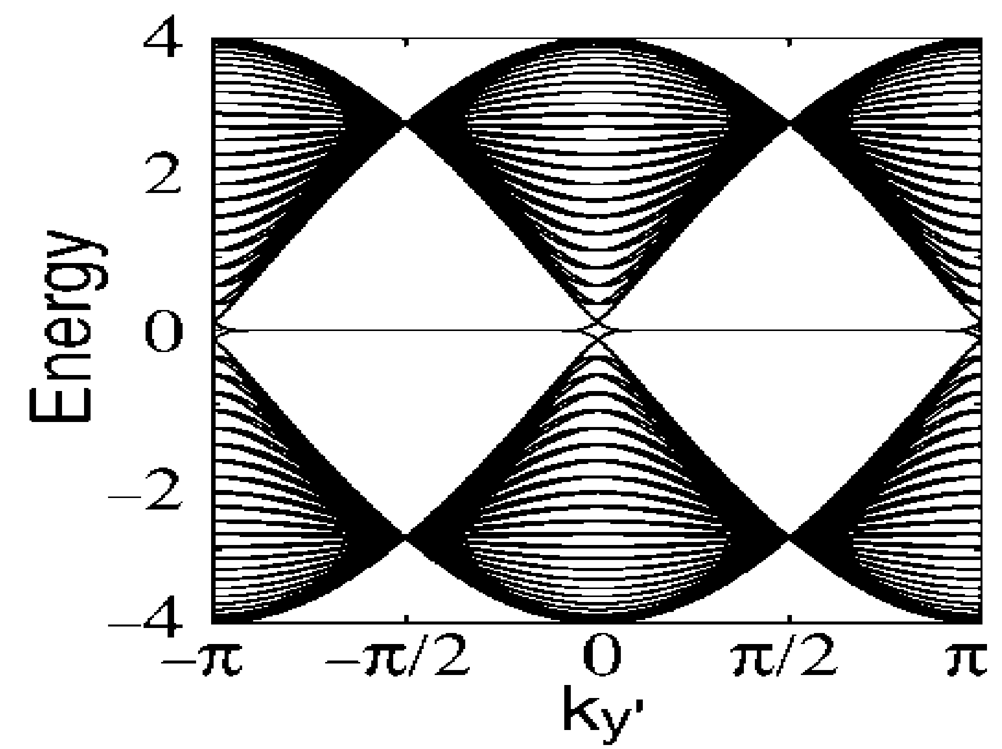
Note: $W_{\mathcal{L}}$ is invariant under path deformation.

Topologically stable point nodes in d_{x2-y2} -wave SCs

- **Bulk-boundary correspondence:** *zero-energy flat* bands on surface of $d_{x^2-y^2}$ SC



Spectrum in slab geometry with (11) face



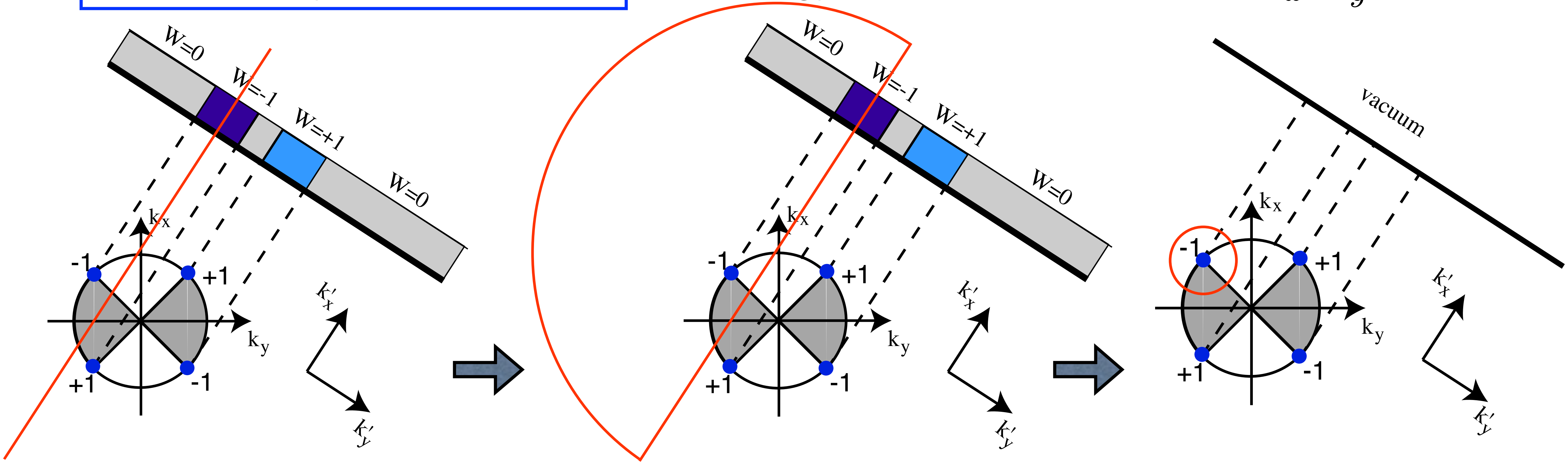
$$W_{\mathcal{L}} = \frac{1}{2\pi i} \oint_{\mathcal{L}} dk_l \ \mathrm{Tr} \left[q^{-1} \partial_{k_l} q \right] = \pm 1$$

- Zero-energy surface states on all edges except the (10) and (01) face.

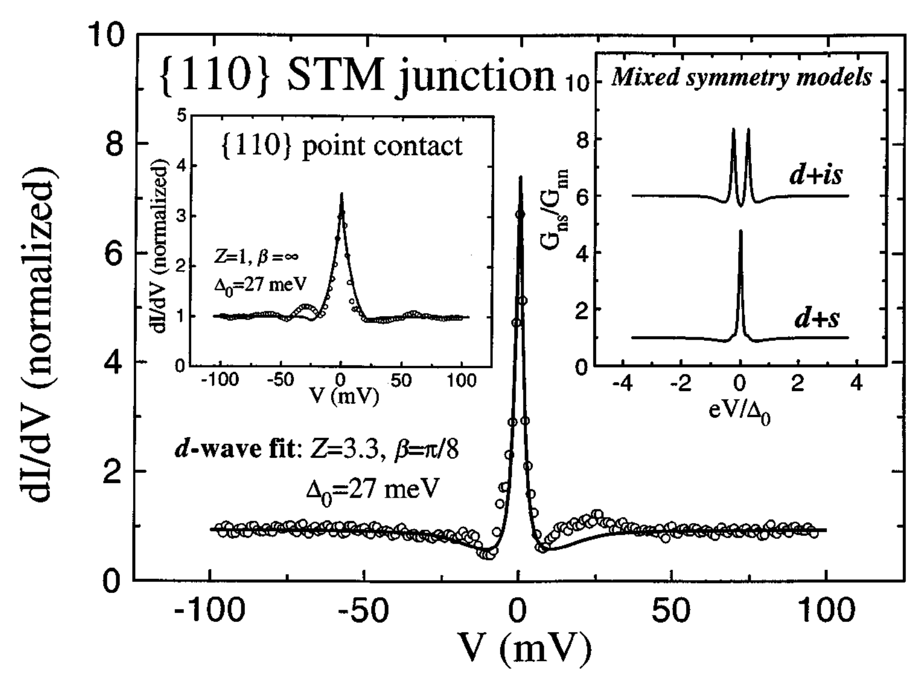
[cf, Hu, PRL 94, Wakabayashi et al. '05]

Topologically stable point nodes in d_{x2-y2} -wave SCs

- Bulk-boundary correspondence: *zero-energy flat* bands on surface of $d_{x^2-y^2}$ SC



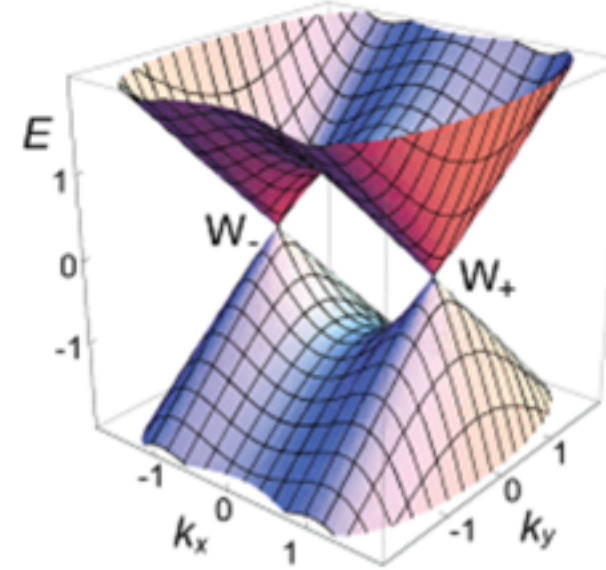
- Experimental observation in high-Tc cuprates:



[Wei et al. PRL '98]

[Kashiwaya et al. '95, Alff et al. '97]

Weyl semi-metal



- Weyl Hamiltonian: $\mathcal{H}(\mathbf{k}) = \mathbf{N}(\mathbf{k}) \cdot \vec{\sigma}$

$$\mathbf{N}(\mathbf{k}) = v_F (k_x, k_y, k_z)^{\mathrm{T}}$$

- No symmetries: $\Rightarrow$ class A

- Topologically stable Weyl points protected by Chern number:

unit skyrmion texture

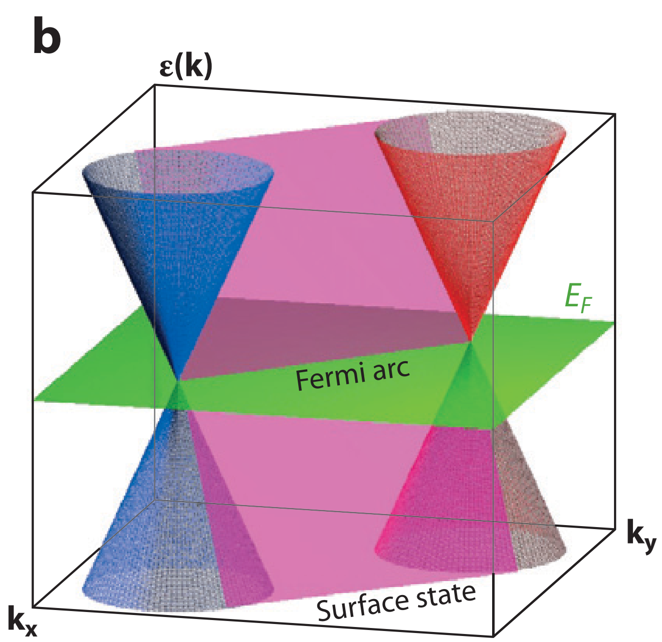
$$N_{\mathcal{C}} = \frac{1}{4\pi} \oint_{\mathcal{C}} d^2\mathbf{k}\, \mathbf{n_k} \cdot [\partial_{k_1} \mathbf{n_k} \times \partial_{k_2} \mathbf{n_k}] = \pm 1 \qquad \mathbf{n_k} = \frac{\mathbf{N}(\mathbf{k})}{|\mathbf{N}(\mathbf{k})|}$$

— Weyl nodes are sources/drains of Berry flux

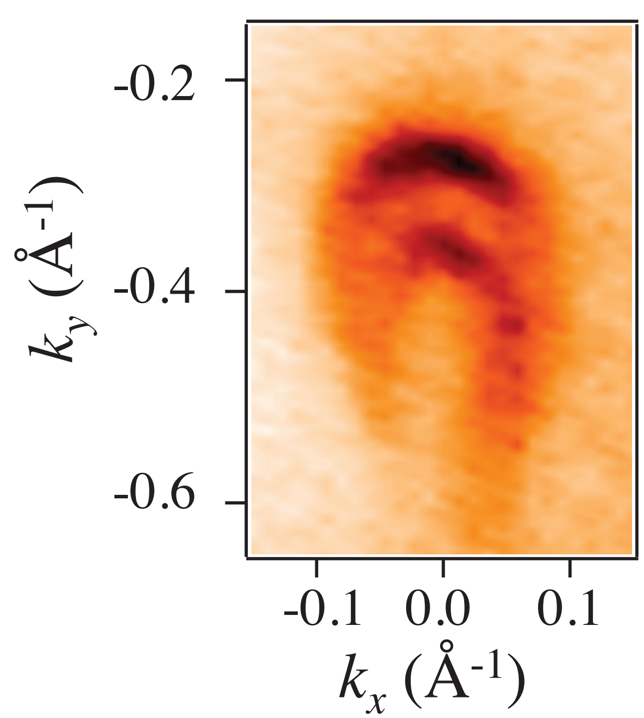
Experimentally realized in: TaAs and NbAs

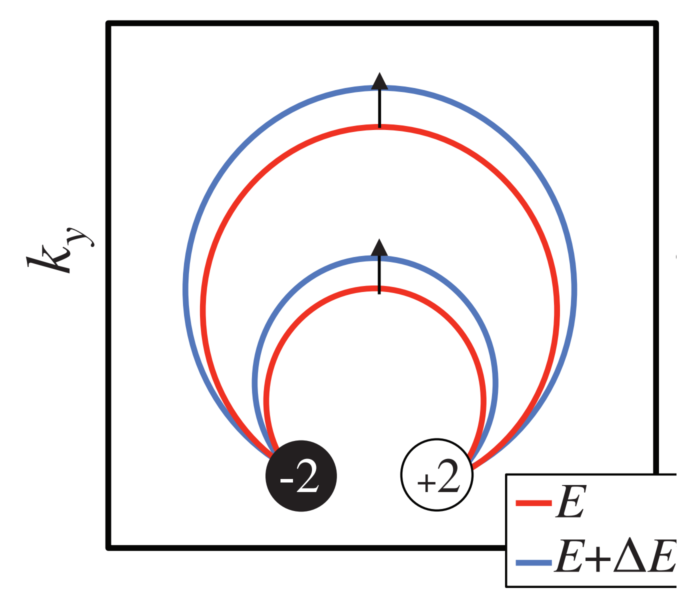
[Su-Yang Xu et al., Science 2015]

- Fermi arc surface states



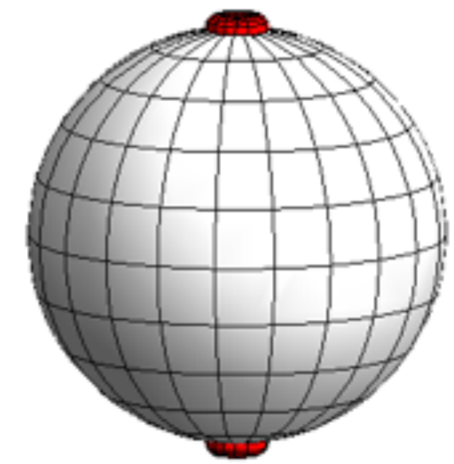
Fermi arcs





Weyl superconductor: 3D chiral p-wave superconductor

aka Anderson-Brinkman-Morel state A phase of ^{3}He



Weyl point

Lattice BdG model: $H_{\mathrm{BdG}} = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}} & c_{-\mathbf{k}} \end{pmatrix} \mathcal{H}_{\mathrm{BdG}} \begin{pmatrix} c_{\mathbf{k}} \\ c^\dagger_{-\mathbf{k}} \end{pmatrix}$

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \mathbf{N}(\mathbf{k}) \cdot \vec{\tau} \qquad \mathbf{N}(\mathbf{k}) = (\Delta_0 k_x/k_F, \Delta_0 k_y/k_F, h_{\mathbf{k}})^{\mathrm{T}}$$

- Particle-hole symmetry: $C^{-1}\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})C = -\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) \quad C = \mathcal{K}\tau_x$

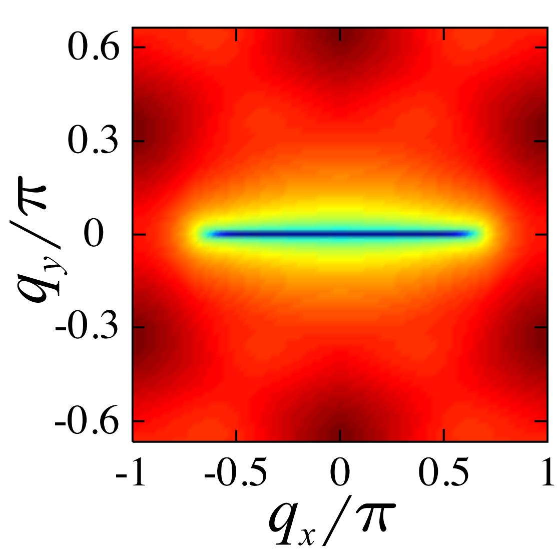
$$C^2 = +1 \quad \Rightarrow \quad \text{class D}$$

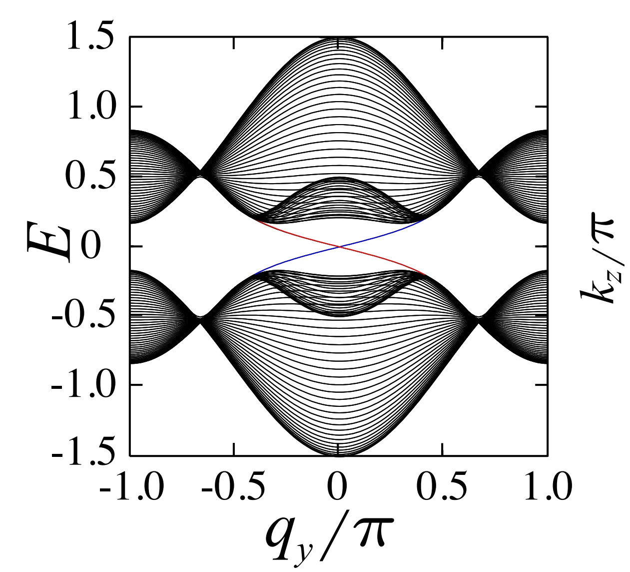
unit skyrmion texture

- Topologically stable Weyl points protected by Chern number: $N_{\mathcal{C}} = \frac{1}{4\pi}\oint_{\mathcal{C}} d^2\mathbf{k}\, \mathbf{n}_{\mathbf{k}} \cdot [\partial_{k_1}\mathbf{n}_{\mathbf{k}} \times \partial_{k_2}\mathbf{n}_{\mathbf{k}}] = \pm 1 \qquad \mathbf{n}_{\mathbf{k}} = \frac{\mathbf{N}(\mathbf{k})}{|\mathbf{N}(\mathbf{k})|}$

- Bulk-boundary correspondence: *surface arc* connecting the projected nodal points

Surface spectrum in slab geometry with (111) face



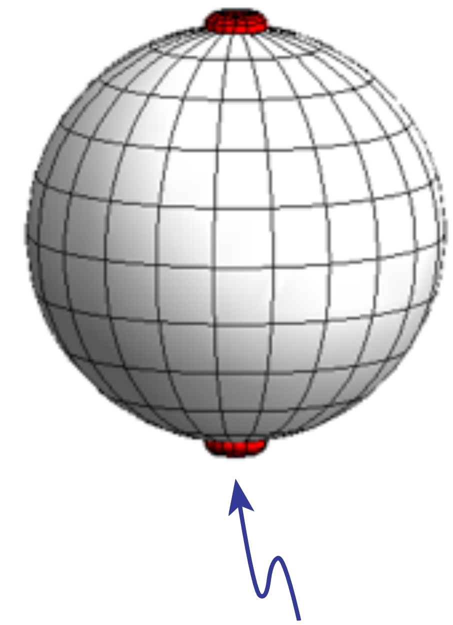


Weyl superconductor: 3D chiral p-wave superconductor

Chern number can be rewritten in terms of Berry curvature $\mathbf{F}(\mathbf{k})$

$$N = \frac{1}{2\pi} \int dk_x dk_y \, F_z(\mathbf{k}) = \pm 1$$

with $\mathbf{F}(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathbf{A}(\mathbf{k})$ and $\mathbf{A}(\mathbf{k}) = i \langle u_-(\mathbf{k}) | \nabla_{\mathbf{k}} | u_-(\mathbf{k}) \rangle$

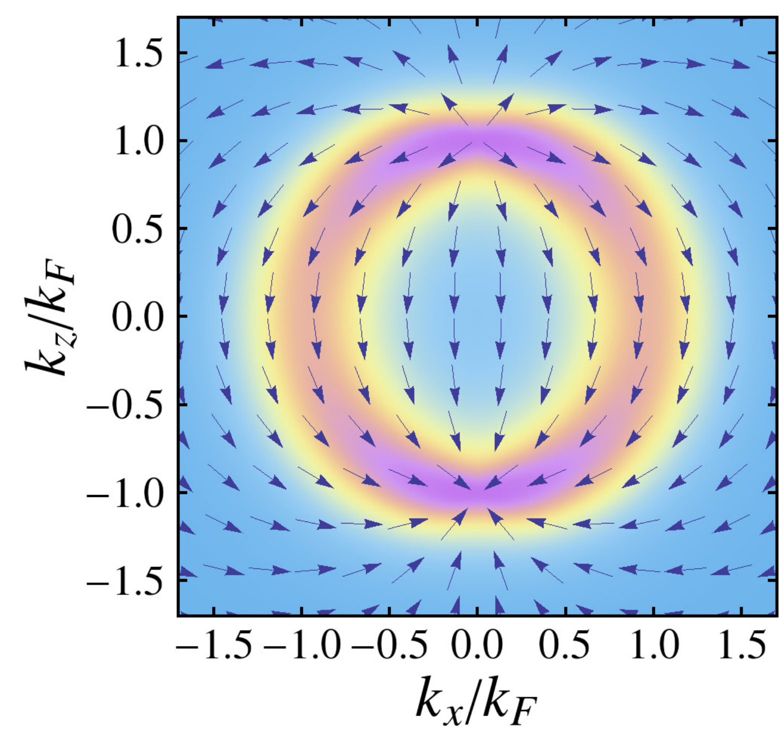


$$F_x(\mathbf{k}) = \frac{\Delta_0^2 k_F^2 k_x k_z}{\mu^2 \left[(k^2 - k_F^2)^2 + \frac{\Delta_0^2}{\mu^2} k_F^2 (k_x^2 + k_y^2) \right]^{\frac{3}{2}}}$$

$$F_y(\mathbf{k}) = \frac{\Delta_0^2 k_F^2 k_y k_z}{\mu^2 \left[(k^2 - k_F^2)^2 + \frac{\Delta_0^2}{\mu^2} k_F^2 (k_x^2 + k_y^2) \right]^{\frac{3}{2}}}$$

$$F_z(\mathbf{k}) = \frac{\Delta_0^2 k_F^2 \left(k_z^2 - k_x^2 - k_y^2 - k_F^2 \right)}{2\mu^2 \left[(k^2 - k_F^2)^2 + \frac{\Delta_0^2}{\mu^2} k_F^2 (k_x^2 + k_y^2) \right]^{\frac{3}{2}}}$$

Berry curvature:



- Weyl nodes are sources and drains of Berry curvature
- Berry flux of 2π flowing from one Weyl node to the other

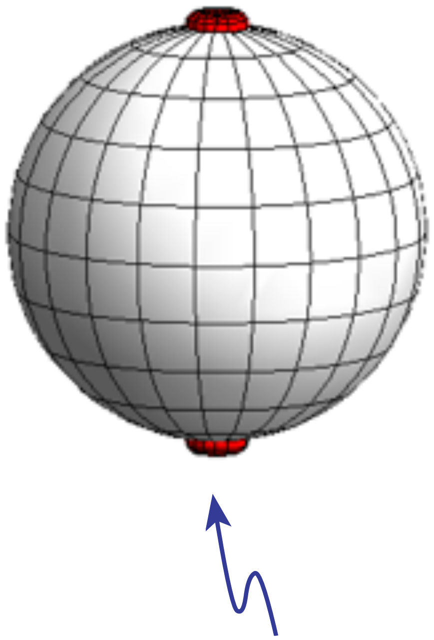
"Double" Weyl superconductor: 3D chiral d-wave SC

possible experimental realizations: URu_2Si_2, SrPtAs

Lattice BdG model: $H_{\mathrm{BdG}} = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} c^{\dagger}_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} \mathcal{H}_{\mathrm{BdG}} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix}$

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \mathbf{N}(\mathbf{k}) \cdot \vec{\tau} \qquad \mathbf{N}(\mathbf{k}) = \left(\Delta_0(k_x^2 - k_y^2)/k_F^2, 2\Delta_0 k_x k_y/k_F^2, h_{\mathbf{k}}\right)$$

$$\text{PHS and SU(2) spin-rotation symmetry} \quad \Rightarrow \quad \text{class C}$$



- Topologically stable double Weyl points protected by Chern number:

$$N_{\mathcal{C}} = \frac{1}{4\pi}\oint_{\mathcal{C}} d^2\mathbf{k}\,\mathbf{n}_{\mathbf{k}} \cdot [\partial_{k_1}\mathbf{n}_{\mathbf{k}} \times \partial_{k_2}\mathbf{n}_{\mathbf{k}}] = \pm 2 \qquad \mathbf{n}_{\mathbf{k}} = \frac{\mathbf{N}(\mathbf{k})}{|\mathbf{N}(\mathbf{k})|}$$

double Weyl points

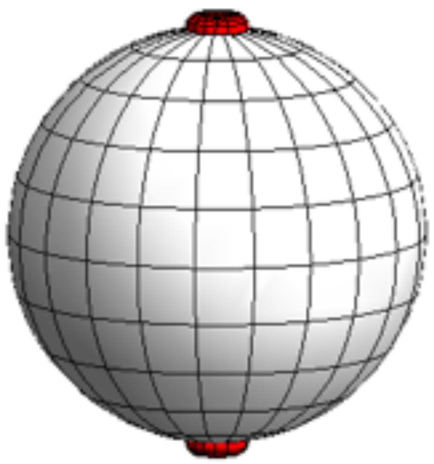
- Bulk-boundary correspondence:
  two spin-degenerate arc surface states

"Double" Weyl superconductor: 3D chiral d-wave SC

Chern number can be reexpressed in terms of Berry curvature $\mathbf{F}(\mathbf{k})$

$$N = \frac{1}{2\pi}\int dk_x dk_y \, F_z(\mathbf{k}) = \pm 2$$

with $\mathbf{F}(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathbf{A}(\mathbf{k})$ and $\mathbf{A}(\mathbf{k}) = i \langle u_-(\mathbf{k}) | \nabla_{\mathbf{k}} | u_-(\mathbf{k}) \rangle$



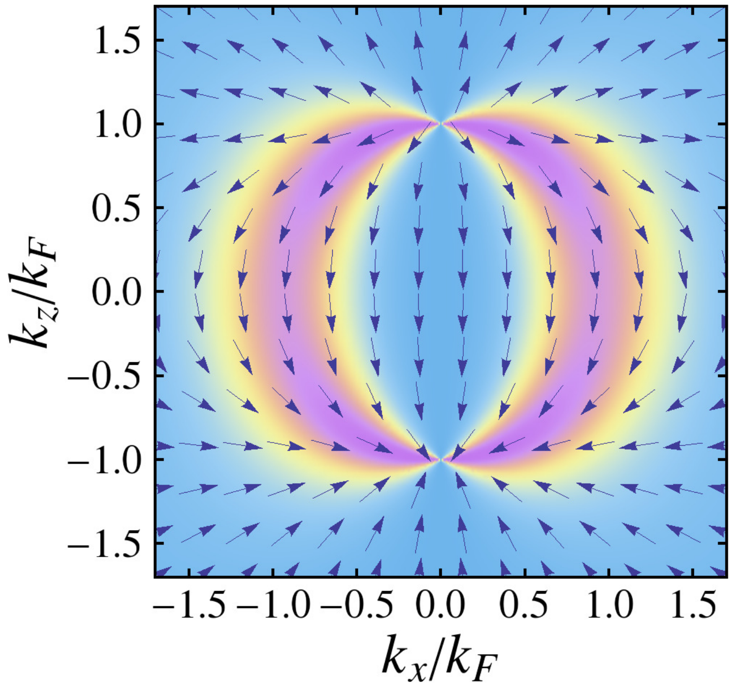
double Weyl points

$$F_x(\mathbf{k}) = \frac{2\Delta_0^2 k_x k_z (k_x^2 + k_y^2)}{\mu^2 \left[(k^2 - k_F^2)^2 + \frac{\Delta_0^2}{\mu^2}(k_x^2 + k_y^2)^2 \right]^{\frac{3}{2}}}$$

$$F_y(\mathbf{k}) = \frac{2\Delta_0^2 k_y k_z (k_x^2 + k_y^2)}{\mu^2 \left[(k^2 - k_F^2)^2 + \frac{\Delta_0^2}{\mu^2}(k_x^2 + k_y^2)^2 \right]^{\frac{3}{2}}}$$

$$F_z(\mathbf{k}) = \frac{2\Delta_0^2 (k_z^2 - k_F^2)(k_x^2 + k_y^2)}{\mu^2 \left[(k^2 - k_F^2)^2 + \frac{\Delta_0^2}{\mu^2}(k_x^2 + k_y^2)^2 \right]^{\frac{3}{2}}}$$

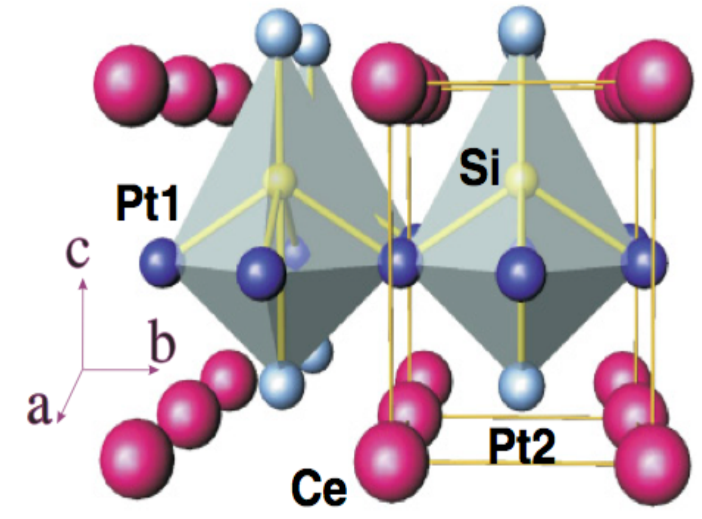
Berry curvature:



- Weyl nodes are double anti-monopoles of the Berry curvature

Nodal non-centrosymmetric superconductors

[E. Bauer et al. PRL '04]



- Lack of inversion causes anti-symmetric SO coupling:

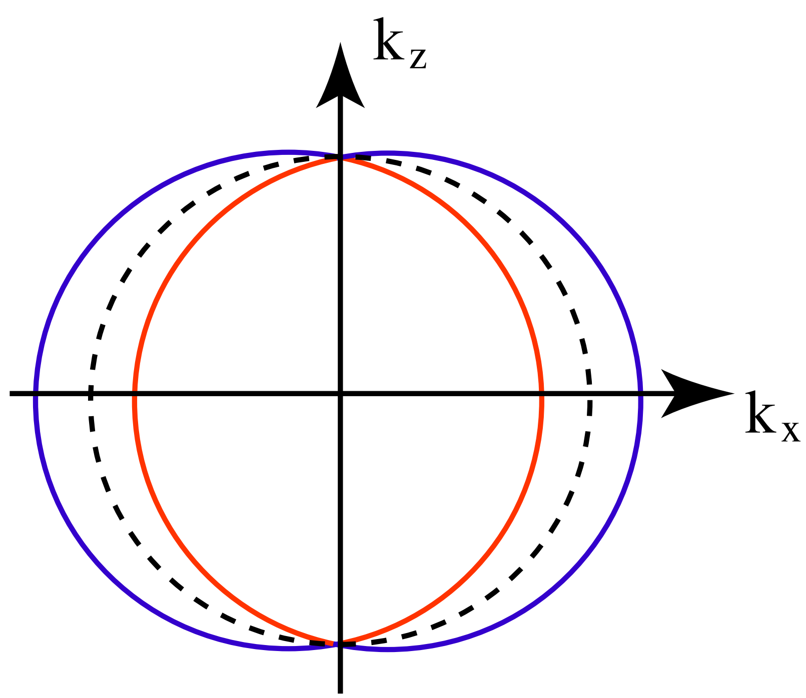
  Normal state: $\mathcal{H} = \sum_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} \sigma_0 + |\mathbf{g}_{\mathbf{k}}| \, \sigma_3 \right) \Psi_{\mathbf{k}}$

  SO coupling for C_{4v} point group: $\boldsymbol{g}_{\boldsymbol{k}} = k_y \hat{\boldsymbol{x}} - k_x \hat{\boldsymbol{y}}$

- Lack of inversion allows for admixture of spin-singlet and spin-triplet pairing components

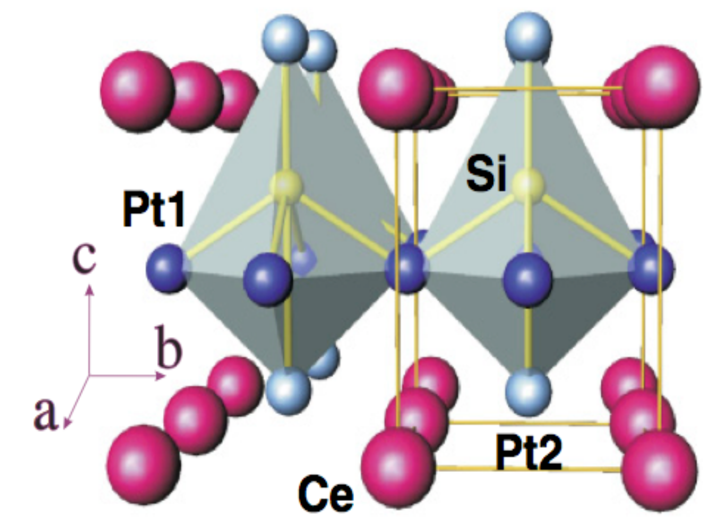
$$\Delta_{\mathbf{k}} = \left(\Delta_{\mathrm{s}} \sigma_0 + \Delta_{\mathrm{t}} \, \mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma} \right) i \sigma_y \qquad (\boldsymbol{g}_{\boldsymbol{k}} \parallel \boldsymbol{d}_{\boldsymbol{k}})$$

Gaps on the two Fermi surfaces: $\boxed{\Delta^{\pm}_{\boldsymbol{k}} = \Delta_s \pm \Delta_p \, |\boldsymbol{d}_{\boldsymbol{k}}|}$



Nodal non-centrosymmetric superconductors

[E. Bauer et al. PRL '04]



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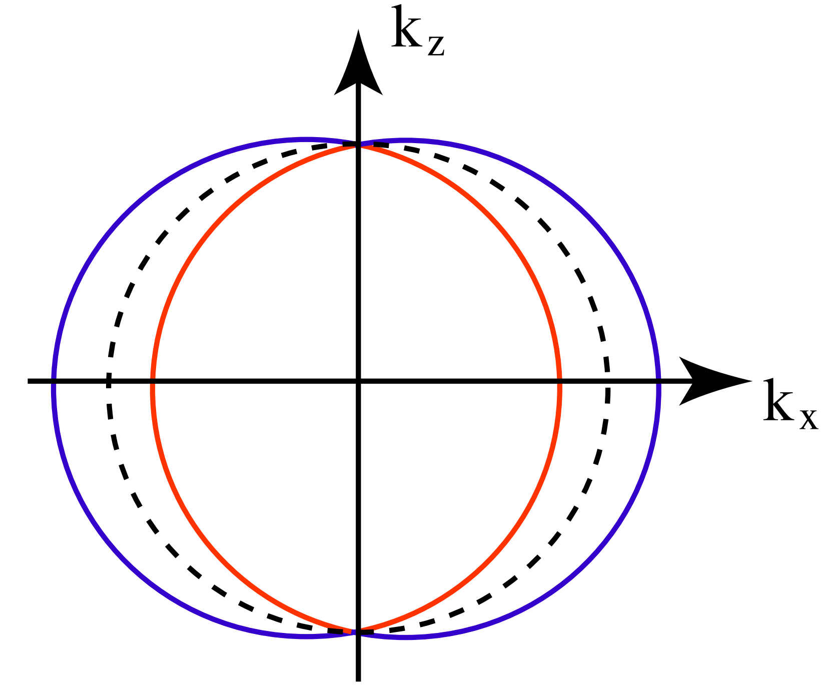
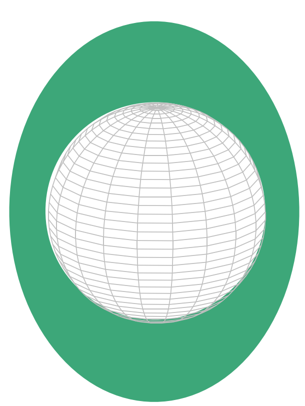


Gaps on the two Fermi surfaces: $\boxed{\Delta_{\boldsymbol{k}}^{\pm} = \Delta_s \pm \Delta_p \, |\boldsymbol{d}_{\boldsymbol{k}}|}$

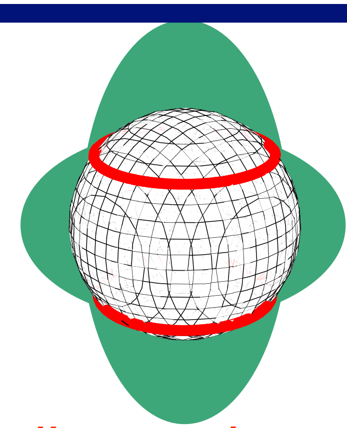
negative helicity FS

$\Delta_s > \Delta_t$



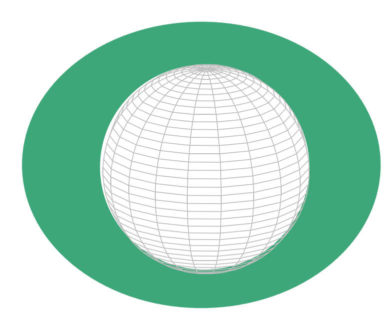
full gap

$\Delta_s \sim \Delta_t$



line nodes

$\Delta_s < \Delta_t$



full gap

Nodal topological superconductors

Consider nodal topological superconductor

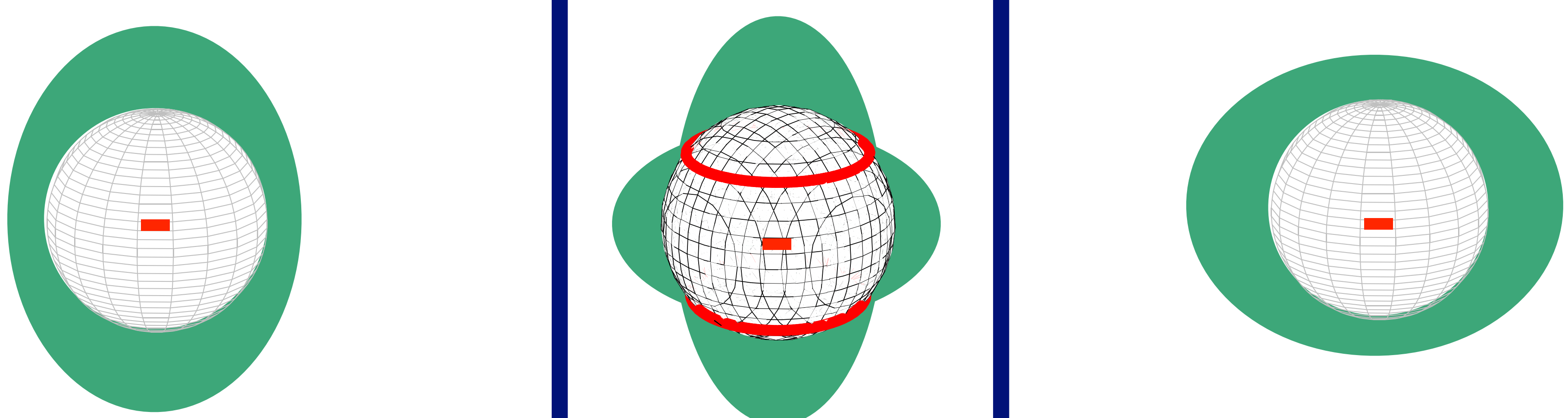
Non-centro SC: $\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \varepsilon_{\mathbf{k}}\sigma_0 + \lambda\, \mathbf{g}_{\mathbf{k}} \cdot \vec{\sigma} & [\Delta_s \sigma_0 + \Delta_t \mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}](i\sigma_y) \\ (-i\sigma_y)[\Delta_s \sigma_0 + \Delta_t \mathbf{d}_{\mathbf{k}} \cdot \vec{\sigma}] & -\varepsilon_{\mathbf{k}}\sigma_0 - \lambda\, \mathbf{g}_{\mathbf{k}} \cdot \vec{\sigma}^* \end{pmatrix}$

Spin-split Fermi surfaces: $\xi^{\pm}_{\mathbf{k}} = \varepsilon_{\mathbf{k}} \pm \lambda |\mathbf{g}_{\mathbf{k}}|$

Gaps on the two Fermi surfaces: $\boxed{\Delta^{\pm}_{\boldsymbol{k}} = \Delta_s \pm \Delta_t\, |\boldsymbol{d}_{\boldsymbol{k}}|}$

$\Delta_s > \Delta_t$ | $\Delta_s \sim \Delta_t$ | $\Delta_s < \Delta_t$

negative helicity Fermi surface



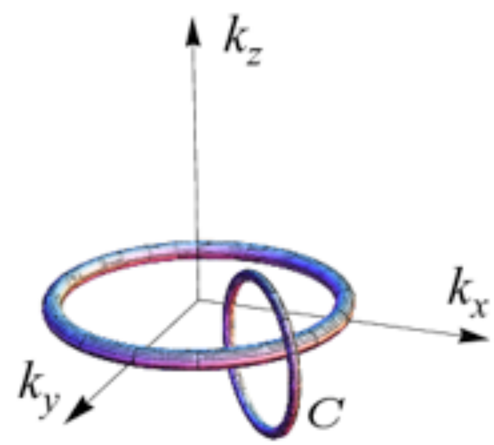
full gap | *line nodes* | *full gap*

$\Delta_s \sim \Delta_t \quad \Longrightarrow$ *nodal lines* on negative helicity FS

Topologically non-trivial

Nodal topological superconductors

▶ Topological characteristics depend on the symmetries of BdG Hamiltonian restricted to contour $\mathcal{C}$.



Classification

	Symmetry			dim		
Class	T	P	S	1	2	3
A	0	0	0	0	$\mathbb{Z}$	0
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	0	0	0
BDI	1	1	1	$\mathbb{Z}$	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	$\mathbb{Z}$	0
CI	1	-1	1	0	0	$\mathbb{Z}$

$$\mathbf{d}_{\mathbf{k}} = (\sin k_x + \sin k_y, \sin k_x + \sin k_y, \sin k_z)^{\mathrm{T}}$$

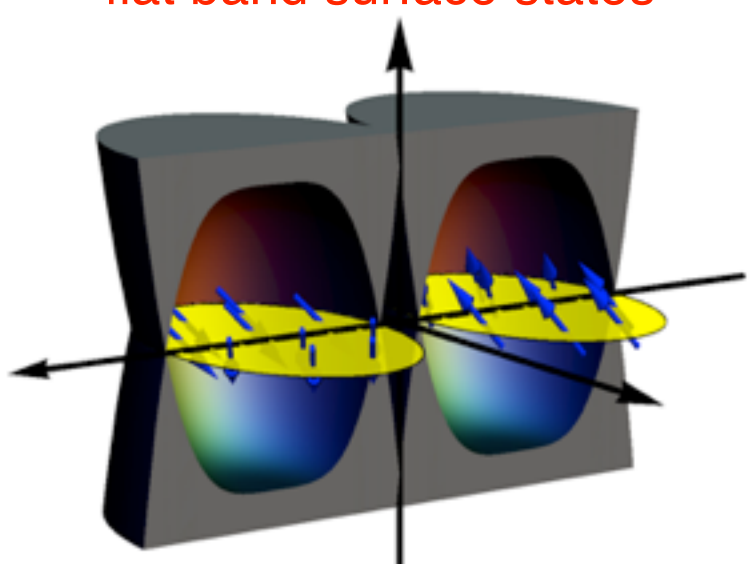
$$\Delta_s \sim \Delta_t$$

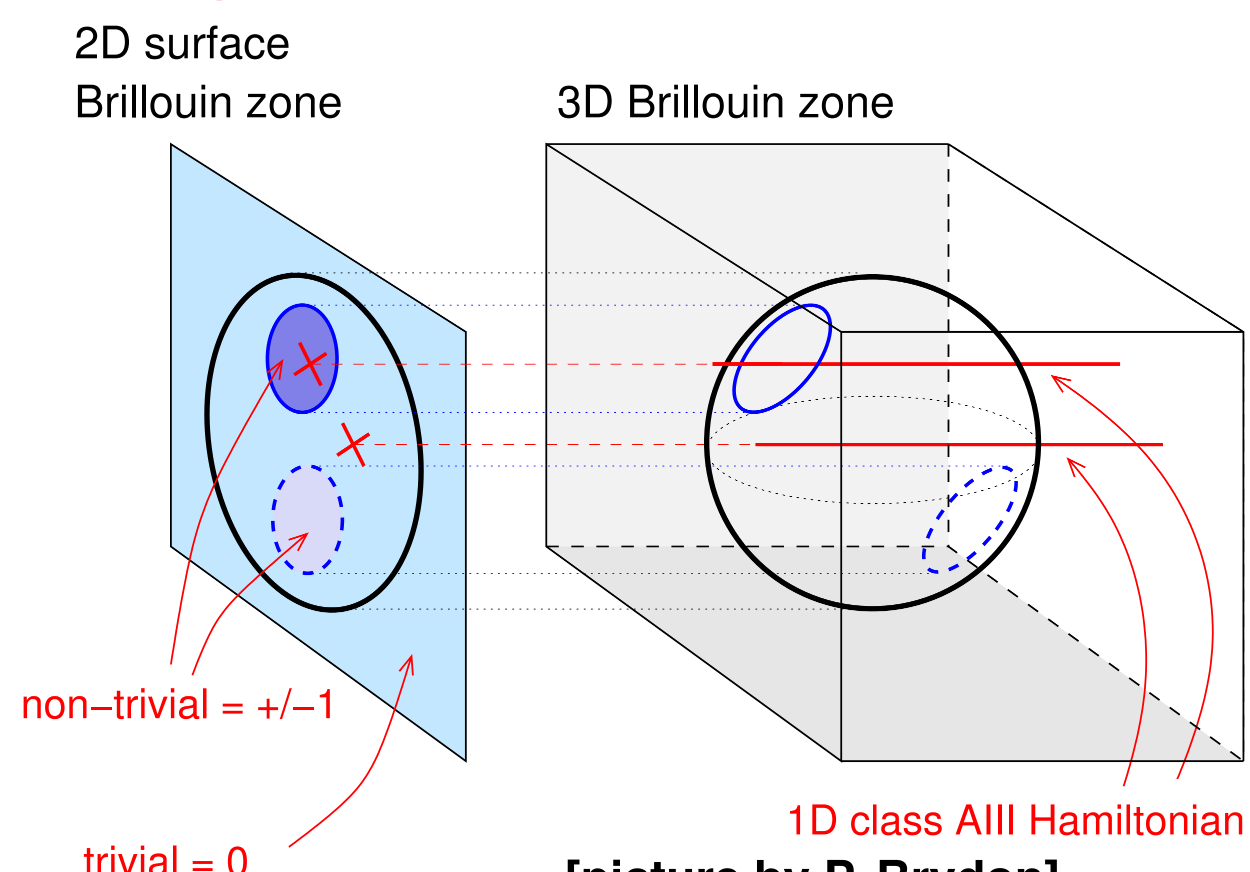
If 1D contour *is not* centrosymmetric: TRS PHS TRS+PHS (chiral sym S)

AIII: 1D Winding number:

$$W_{\mathcal{C}} = \frac{1}{2\pi} \oint_{\mathcal{C}} dk_l \, \partial_{k_l} \left[\arg(\xi^-_{\mathbf{k}} + i\Delta^-_{\mathbf{k}}) \right]$$

flat band surface states





[picture by P. Brydon]

Classification of topological nodal superconductors

in terms of global symmetries (TRS, PHS, SLS)

Zhao, Wang, PRL 2013

Matsuura, Schnyder, et al. NJP 2013

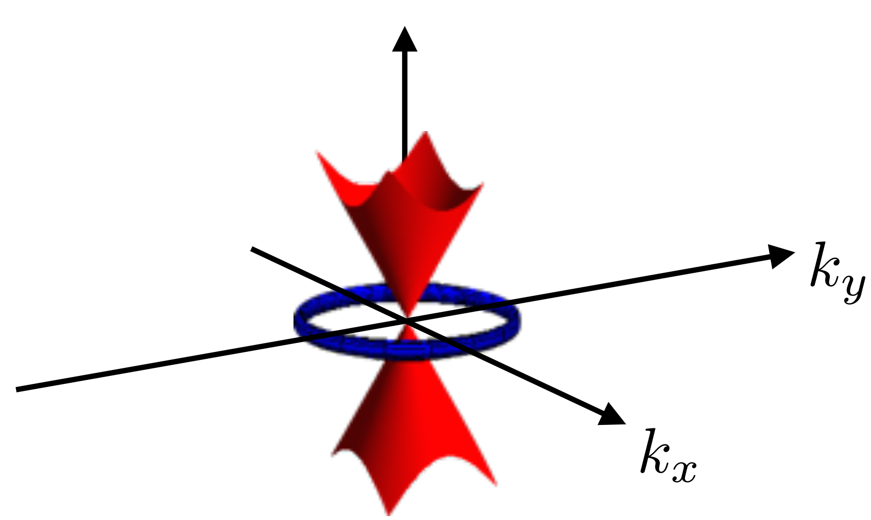
▶ Classification of topological nodal semimetals and superconductors depends on:

- symmetry of Hamiltonian (TRS, PHS, SLS)

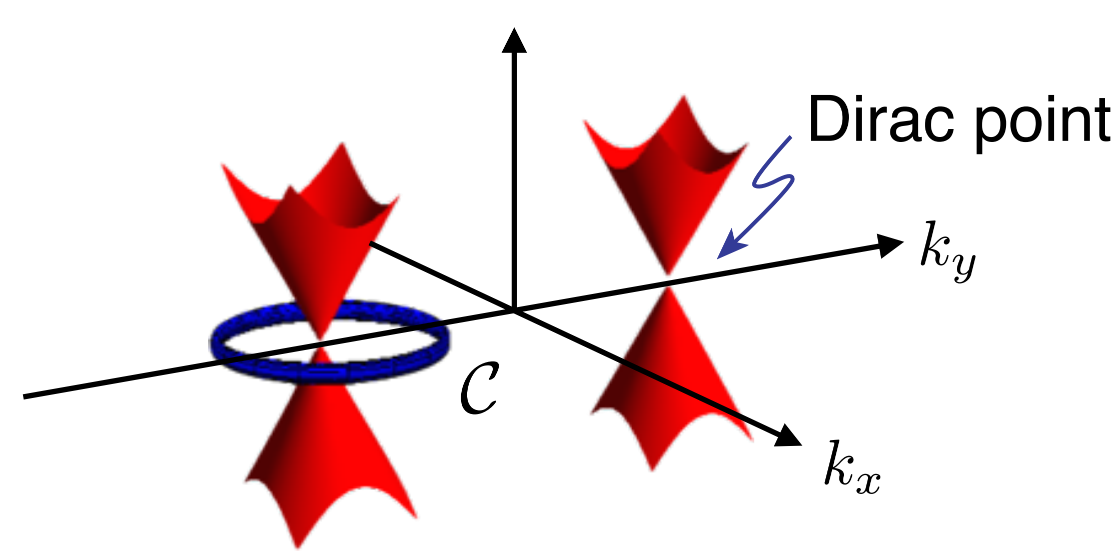
  $\Rightarrow$ **symmetry classes of ten-fold way**

- co-dimension $p = d - d_{\mathrm{FS}}$ of Fermi surface (d_{FS}: dimension of Fermi surface)
- how Fermi surface transforms under global symmetries

(i) Fermi surface is *invariant* under global symmetries

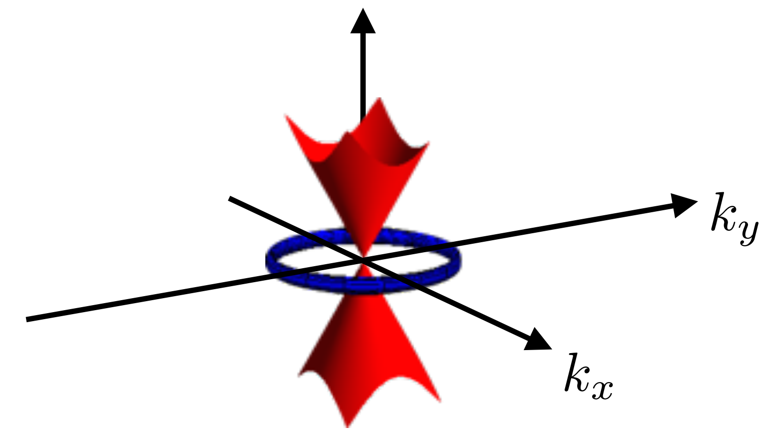


(ii) Fermi surfaces *pairwise related* by global symmetries



Gapless Dirac Hamiltonians

(i) Fermi surface is *invariant* under global symmetries



$$\mathcal{H}(k) = \sum_{i=1}^{d} k_i \gamma_i \qquad\qquad E_{\pm} = \pm\sqrt{\sum_{i=1}^{d} k_i^2}$$

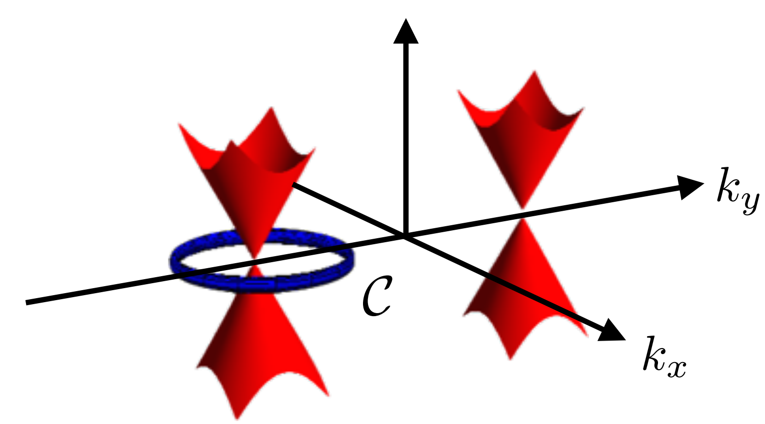
? are there symmetry preserving mass terms $M\gamma_{d+1}$
that open up a gap in the spectrum?

$$\{\gamma_{d+1}, \gamma_i\} = 0 \quad i = 0, 1, \cdots 2 \qquad\qquad E_{\pm} = \pm\sqrt{M^2 + \sum_{i=1}^{d} k_i^2}$$

NO: topologically non-trivial **YES:** topologically trivial

– to distinguish between $\mathbb{Z}_2$ and $\mathbb{Z}$ classification
consider doubled version of Hamiltonian

(ii) Fermi surfaces *pairwise related* by global symmetries



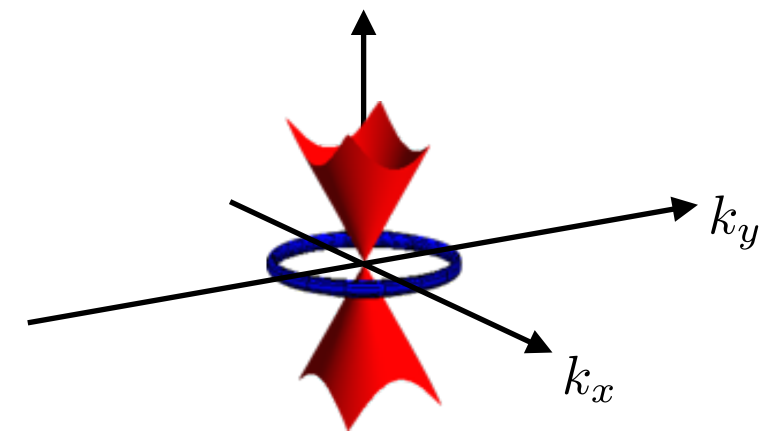
$$\mathcal{H}(k) = \sum_{i=1}^{p-1} \sin k_i \gamma_i + \left(1 - p + \sum_{i=1}^{p} \cos k_i\right)\gamma_0$$

– semi-metal with $(d-p)$-dimensional Fermi surface

Classification of topological nodal superconductors

(i) Fermi surface is *invariant* under global symmetries

$$p = d - d_{\rm FS}$$



classification of *Fermi points* with $d_{\rm FS} = 0$ in **d** dimensions ⟷ classification of *fully gapped* topological materials in **d+1** dimensions

at high-sym. point	T	C	S	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0
BDI	+1	+1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$
D	0	+1	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2{}^{\dagger,\S}$
DIII	−1	+1	1	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$
AII	−1	0	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
CII	−1	−1	1	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
C	0	−1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0
CI	+1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0

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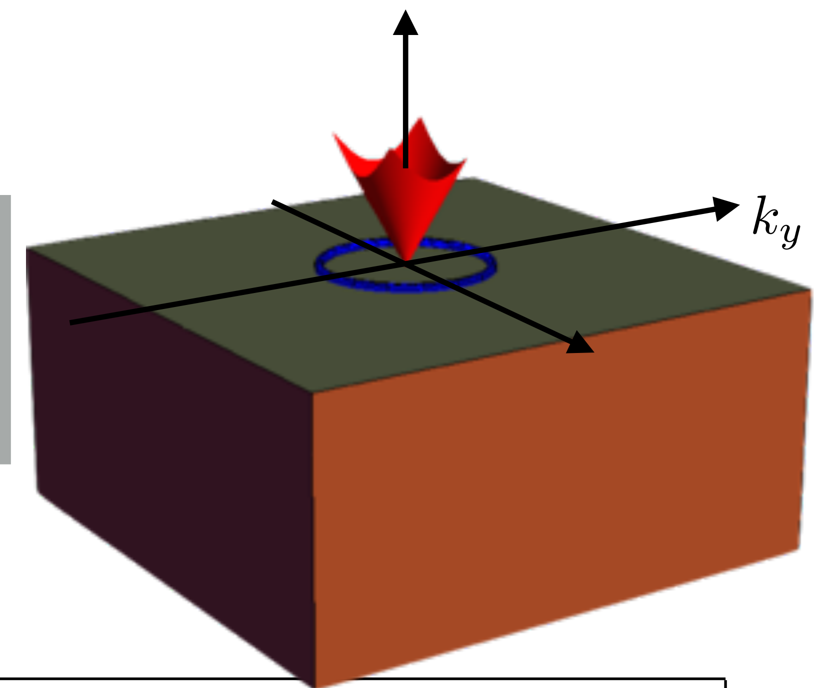
Classification of topological nodal superconductors

(i) Fermi surface is *invariant* under global symmetries

$$p = d - d_{\mathrm{FS}}$$



classification of *Fermi points* with $d_{\mathrm{FS}} = 0$ in **d** dimensions	⟷	classification of *fully gapped* topological materials in **d+1** dimensions

at high-sym. point	T	C	S	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0
BDI	+1	+1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$
D	0	+1	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	${\mathbb{Z}_2}^{\dagger,\S}$
DIII	−1	+1	1	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$
AII	−1	0	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
CII	−1	−1	1	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
C	0	−1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0
CI	+1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0

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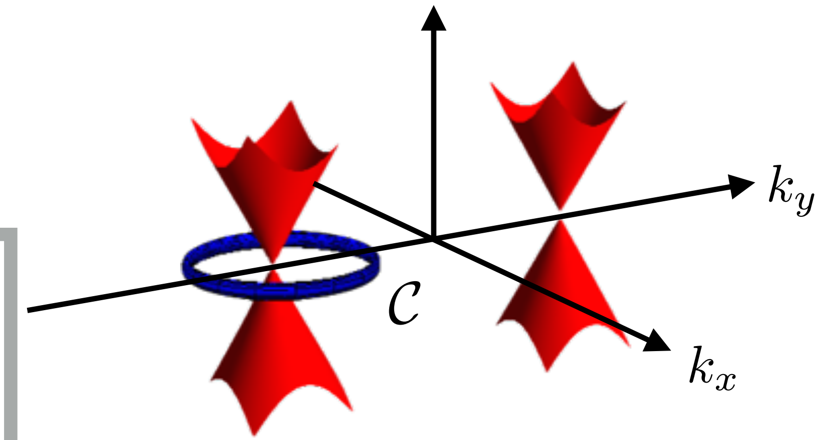
Classification of topological nodal superconductors

(ii) Fermi surfaces *pairwise related* by global global symmetries

$$p = d - d_{\mathrm{FS}}$$



classification of *Fermi points* with $d_{\mathrm{FS}} = 0$ in **d** dimensions ⟷ classification of *fully gapped* topological materials in **d-1** dimensions

off high-sym. point	T	C	S	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$
BDI	+1	+1	1	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$
D	0	+1	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2{}^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−1	+1	1	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0
CII	−1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0
C	0	−1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0
CI	+1	−1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$

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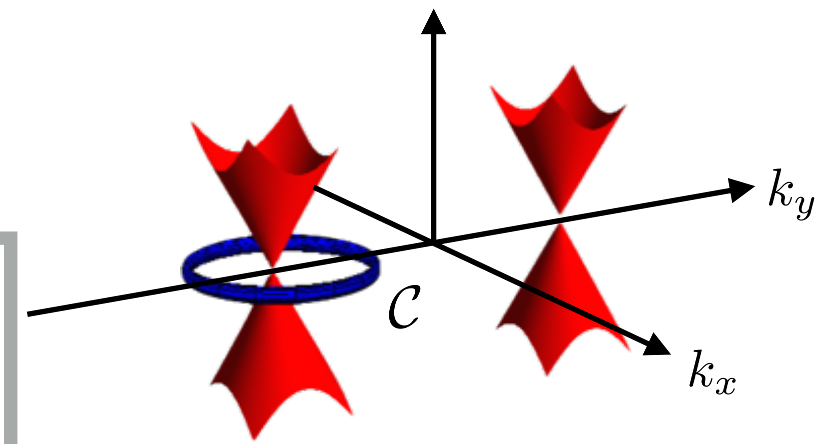
Classification of topological nodal superconductors

(ii) Fermi surfaces *pairwise related* by global global symmetries

$$p = d - d_{\mathrm{FS}}$$



classification of *Fermi points* with $d_{\mathrm{FS}} = 0$ in **d** dimensions ⟷ classification of *fully gapped* topological materials in **d-1** dimensions

off high-sym. point	T	C	S	p=1	p=2	p=3	p=4	[illegible]	[illegible]	[illegible]	p=8
A	0	0	0	$\mathbb{Z}$	0	$\boxed{\mathbb{Z}}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\boxed{\mathbb{Z}}$	0	$\mathbb{Z}$	[illegible]	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$
BDI	+1	+1	1	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	[illegible]	[illegible]	[illegible]	[illegible]
D	0	+1	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2{}^{\dagger,\S}$	$\boxed{\mathbb{Z}}$	0	0	0	$2\mathbb{Z}$	0
DIII	−1	+1	1	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0	0	0
CII	−1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	[illegible]	[illegible]	[illegible]	0
C	0	−1	0	0	0	$\boxed{2\mathbb{Z}}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$	0
CI	+1	−1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}_2^{\dagger,\S}$	$\mathbb{Z}$

Weyl semi-metal

NCS

Weyl superconductor

chiral d-wave SC

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